Testing Endogeneity with High Dimensional Covariates*

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Abstract

Modern, high dimensional data has renewed investigation on instrumental variables (IV) analysis, primary focusing on estimation of the included endogenous variable under sparsity and little attention towards specification tests. This paper studies in high dimensions the Durbin-Wu-Hausman (DWH) test, a popular specification test for endogeneity in IV regression. We show, surprisingly, that the DWH test maintains its size in high dimensions, but at an expense in power. We propose a new test that remedies this issue and has better power than the DWH test. Simulation studies reveal that our test achieves near-oracle performance to detect endogeneity.

JEL classification: C12; C36
Keywords: Endogeneity test; Durbin-Wu-Hausman test; High dimension; Instrumental variable; Power function

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1 Introduction

1.1 Endogeneity Testing with High Dimensional Data

Recent growth in both the size and dimension of the data has led to a resurgence in analyzing instrumental variables (IV) regression in high dimensional settings (Belloni et al., 2012, 2013, 2011a; Chernozhukov et al., 2014, 2015; Fan and Liao, 2014; Gautier and Tsybakov, 2011) where the number of regression parameters, especially those associated with exogenous covariates, is growing with, and may exceed, the sample size. The primary focus in this work has been providing tools for estimation and inference of a single endogenous variable's effect on the outcome under some low-dimensional structural assumptions on the structural parameters associated with the instruments and the covariates, such as sparsity. (Belloni et al., 2012, 2013, 2011a; Chernozhukov et al., 2014, 2015; Gautier and Tsybakov, 2011). Unfortunately, this line of work has generally not focused on specification tests in the high dimensional IV settings.

The main goal of this paper is to study the high dimensional behavior of one of the most common specification test in IV regression, the test for endogeneity. Historically, the most widely used test for endogeneity is the Durbin-Wu-Hausman test (Durbin, 1954; Hausman, 1978; Wu, 1973), hereafter called the DWH test and is widely implemented in software, such as ivreg2 in Stata (Baum et al., 2007). The DWH test detects the presence of endogeneity in the structural model by studying the difference between the ordinary least squares (OLS) estimate of the structural parameters in the IV regression to that of the two-stage least squares (TSLS) under the null hypothesis of no endogeneity; see Section 2.3 for the exact characterization of the DWH test. In non-high dimensional settings, the primary requirements for the DWH test to correctly control Type I error are having instruments that are (i) strongly associated with the included endogenous variable, often called strong instruments, and (ii) exogenous to the structural errors, often referred to as

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1In the paper, we use the term “high dimensional setting” more broadly where the number of parameters is growing with the sample size; see Sections 3 and 4.3 for details and examples. Note that the modern usage of the term “high dimensional setting” where the sample size exceeds the parameter is one case of this broad setting.

2The term exogeneity is sometimes used in the IV literature to encompass two assumptions, (a) inde-
valid instruments (Murray, 2006). For example, when instruments are not strong, Staiger and Stock (1997) showed that the DWH test that used the TSLS estimator for variance, which is attributed to Durbin (1954) and Wu (1973), had distorted size under the null hypothesis while the DWH test that used the OLS estimator for variance, which is attributed to Hausman (1978), had proper size. When instruments are invalid, which is perhaps a bigger concern in practice (Conley et al., 2012; Murray, 2006), the DWH test will usually fail because the TSLS estimator is inconsistent under the null hypothesis; see the Supplementary materials for a simple theoretical justification of this phenomena. Indeed, some recent work with high dimensional data has advocated conditioning on many, possibly high dimensional, exogenous covariates to make instruments more plausibly valid (Belloni et al., 2012; Chernozhukov et al., 2015). However, while adding more covariates can potentially make instruments more plausibly valid, it is unclear what price one has to pay with respect to the performance of the specification tests like the DWH test.

1.2 Prior Work and Contribution

Prior work in analyzing the DWH test in instrumental variables is diverse. Estimation and inference under weak and/or many instruments are well documented (Andrews et al., 2007; Bekker, 1994; Bound et al., 1995; Chao and Swanson, 2005; Dufour, 1997; Han and Phillips, 2006; Hansen et al., 2008; Kleibergen, 2002; Moreira, 2003; Morimune, 1983; Nelson and Startz, 1990; Newey and Windmeijer, 2005; Staiger and Stock, 1997; Stock and Yogo, 2005; Wang and Zivot, 1998; Zivot et al., 1998). In particular, when the instruments are weak, the behavior of the DWH test under the null depends on the variance estimate. Dependence of the IVs to the disturbances in the structural model and (b) IVs having no direct effect on the outcome, sometimes referred to as the exclusion restriction (Angrist et al., 1996; Holland, 1988; Imbens and Angrist, 1994). As such, an instrument that is perfectly randomized from a randomized experiment may not be exogenous in the sense that while the instrument is independent to any structural error terms, the instrument may still have a direct effect on the outcome.

3 For example, in Section 7 of the empirical example of Belloni et al. (2012), the authors study the effect of federal appellate court decisions on economic outcomes by using the random assignment of judges to decide appellate cases. They state that “once the distribution of characteristics” of federal circuit court judges in a given circuit-year is controlled for, “the realized characteristics of the randomly assigned three-judge panel should be unrelated to other factors besides judicial decisions that may be related to economic outcomes” (page 2405). More broadly, in empirical practice, adding covariates to make IVs more plausibly valid is commonplace; see Card (1999), Cawley et al. (2013), and Kosec (2014) for examples as well as review papers in epidemiology and causal inference by Hernán and Robins (2006) and Baiocchi et al. (2014).
Some recent work extends the specification test to handle growing number of instruments (Chao et al., 2014; Hahn and Hausman, 2002a; Lee and Okui, 2012). Other recent works extend specification tests based on overidentification (Hahn and Hausman, 2002b; Hausman et al., 2005). Fan et al. (2015) considers testing endogeneity in high dimensional setting in non-IV settings and approximate the null distribution by bootstrap, whereas the distribution under the alternative is not identifiable. None of these works have characterized the properties of the DWH test used in IV regression under high dimensional settings.

Our main contributions are two-fold. First, we characterize the behavior of the popular DWH test in high dimensions. The theoretical analysis reveals that the DWH test actually controls Type I error at the correct level in high dimension, but pays a significant price with respect to power, especially for small to moderate degrees of endogeneity; we also confirm our finding numerically with a simulation study of the empirical power of the DWH test. Our finding also suggests that when empirical work conditions on a large number of covariates for either better model specification or to make instruments more plausibly valid, the empiricist pays a price in power reduction of the DWH test. Second, we remedy the low power of the DWH test by presenting a simple and improved endogeneity test that is robust to high dimensional covariates and/or instruments and that also works in settings where the number of structural parameters is allowed to exceed the sample size. In particular, our new endogeneity test modifies popular estimators for reduced-form models that are well-behaved, such as OLS or variants of Lasso (see Section 4.1 for details) by using a hard thresholding procedure. Also, we briefly discuss an extension of our endogeneity test to incorporate invalid instruments, especially when many covariates are conditioned upon to avoid invalid IVs.

We conclude the paper with simulation studies comparing the performance of our new test with the usual DWH test. We find that our test has the desired size and has better power than the DWH test for all degrees of endogeneity and reaches near-optimal performance. In the supplementary materials, we also present technical proofs and extended simulation studies that further examine the power of our test.
2 Instrumental Variables Regression and the DWH Test

2.1 Notation

For any $p$ dimensional vector $v$, the $j$th element is denoted as $v_j$. Let $\|v\|_1$, $\|v\|_2$, and $\|v\|_{\infty}$ denote the $1$, $2$ and $\infty$-norms, respectively. Let $\|v\|_0$ denote the number of non-zero elements in $v$ and let $\text{supp}(v) = \{j : v_j \neq 0\} \subseteq \{1, \ldots, p\}$. For any $n$ by $p$ matrix $M$, denote the $i$th row and $j$th column entry as $M_{ij}$, the $i$th row vector as $M_i$, the $j$th column vector as $M_j$, and $M'$ as the transpose of $M$. Also, given any $n$ by $p$ matrix $M$ with sets $I \subseteq \{1, \ldots, n\}$ and $J \subseteq \{1, \ldots, p\}$ denote $M_{IJ}$ as the submatrix of $M$ consisting of rows specified by the set $I$ and columns specified by the set $J$, $M_I$ as the submatrix of $M$ consisting of rows specified by the set $I$ and all columns, and $M_{.J}$ as the submatrix of $M$ consisting of all rows and columns specified by the set $J$. Also, for any $n \times p$ full-rank matrix $M$, define the orthogonal projection matrices $P_M = M(M'M)^{-1}M'$ and $P_M^\perp = I - M(M'M)^{-1}M'$ where $P_M + P_M^\perp = I$ and $I$ is an identity matrix. For a $p \times p$ matrix $\Lambda$, $\Lambda > 0$ denotes that $\Lambda$ is a positive definite matrix. For any $p \times p$ positive definite $\Lambda$ and set $J \subseteq \{1, \ldots, p\}$, let $\Lambda_{J|J^C} = \Lambda_{JJ} - \Lambda_{JJ^C} \Lambda_{J^CJ}^{-1} \Lambda_{J^CJ}$ denote the submatrix $\Lambda_{JJ}$ adjusted for the columns in the complement of the set $J$, $J^C$.

For a sequence of random variables $X_n$ indexed by $n$, we use $X_n \overset{p}{\to} X$ to represent that $X_n$ converges to $X$ in probability. For a sequence of random variables $X_n$ and numbers $a_n$, we define $X_n = o_p(a_n)$ if $X_n/a_n$ converges to zero in probability and $X_n = O_p(a_n)$ if for every $c_0 > 0$, there exists a finite constant $C_0$ such that $P(|X_n/a_n| \geq C_0) \leq c_0$. For any two sequences of numbers $a_n$ and $b_n$, we will write $b_n \ll a_n$ if $\lim\sup b_n/a_n = 0$.

For notational convenience, for any $\alpha$, $0 < \alpha < 1$, let $\Phi$ and $z_{\alpha/2}$ denote, respectively, the cumulative distribution function and $\alpha/2$ quantile of a standard normal distribution. Also, for any $B \in \mathbb{R}$, we define the function $G(\alpha, B)$ to be the tail probabilities of a normal distribution shifted by $B$, i.e.

\[
G(\alpha, B) = 1 - \Phi(z_{\alpha/2} - B) + \Phi(-z_{\alpha/2} - B). \quad (1)
\]
We also denote $\chi^2_\alpha(d)$ to be the $1 - \alpha$ quantile of the Chi-squared distribution with $d$ degrees of freedom.

### 2.2 Model and Definitions

Suppose we have $n$ individuals where for each individual $i = 1, \ldots, n$, we measure the outcome $Y_i$, the included endogenous variable $D_i$, $p_z$ candidate instruments $Z_i'$, and $p_x$ exogenous covariates $X_i'$ in an i.i.d. fashion. We denote $W_i'$ to be concatenated vector of $Z_i'$ and $X_i'$ with dimension $p = p_z + p_x$. The columns of the matrix $W$ are indexed by the $I = \{1, \ldots, p_z\}$ which consists of all the $p_z$ candidate instruments and the set $I^C = \{p_z + 1, \ldots, p\}$ which consists of the $p_x$ covariates. The variables $(Y_i, D_i, Z_i, X_i)$ are governed by the following structural model

$$ Y_i = D_i \beta + X_i' \phi + \delta_i, \quad E(\delta_i \mid Z_i, X_i) = 0 \quad (2) $$

$$ D_i = Z_i' \gamma + X_i' \psi + \epsilon_i, \quad E(\epsilon_i \mid Z_i, X_i) = 0 \quad (3) $$

where $\beta, \phi, \gamma$, and $\psi$ are unknown parameters in the model and without loss of generality, we assume the variables are centered to mean zero\(^4\). The random disturbance terms $\delta_i$ and $\epsilon_i$ are independent of $(Z_i, X_i)$ and, for simplicity, are assumed to be bivariate normal. Let the population covariance matrix of $(\delta_i, \epsilon_i)$ be $\Sigma$, with $\Sigma_{11} = \text{Var}(\delta_i \mid Z_i, X_i)$, $\Sigma_{22} = \text{Var}(\epsilon_i \mid Z_i, X_i)$, and $\Sigma_{12} = \Sigma_{21} = \text{Cov}(\delta_i, \epsilon_i \mid Z_i, X_i)$. Let the second order moments of $W_i$ be $\Lambda = \mathbf{E}(W_i W_i')$ and let $\Lambda_{\mathcal{I} \mathcal{I}^c}$ denote the adjusted covariance of $W_i$. Let $\omega$ represent all the parameters $\omega = (\beta, \pi, \phi, \gamma, \psi, \Sigma)$ from the parameter space $\omega \in \Omega = \{\mathbb{R} \otimes \mathbb{R}^{p_z} \otimes \mathbb{R}^{p_x} \otimes \mathbb{R}^{p_z} \otimes \mathbb{R}^{p_x} \otimes \Sigma \succ 0\}$. Finally, we denote $s_{x2} = \|\phi\|_0$, $s_{x1} = \|\gamma\|_0$, $s_{x1} = \|\psi\|_0$ and $s = \max\{s_{x2}, s_{x1}, s_{x1}\}$.

We also define relevant and irrelevant instruments. This is, in many ways, equivalent to the notion that the instruments $Z_i$ are associated with the endogenous variable $D_i$, except we use the support of a vector to define the instruments’ association to the endogenous variable; see Breusch et al. (1999); Hall and Peixe (2003), and Cheng and Liao (2015) for details.

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\(^4\)The mean-centering is equivalent to adding a constant 1 term (i.e. intercept term) in $X_i'$; see Section 1.4 of Davidson and MacKinnon (1993) for details.
some examples in the literature of defining relevant and irrelevant instruments based on
the support of a parameter.

**Definition 1.** Suppose we have \( p_z \) instruments along with the model (3). We say that instrument \( j = 1, \ldots, p_z \) is relevant if \( \gamma_j \neq 0 \) and irrelevant if \( \gamma_j = 0 \). Let \( S \subseteq I \) denote the set of relevant IVs.

Finally, for the set of relevant IVs \( S \), we define the concentration parameter, a common measure of instrument strength,

\[
C(S) = \frac{\gamma_S \Lambda_{S|S^C} \gamma_S}{|S| \Sigma_{22}}.
\]  

(4)

If all the instruments were relevant, then \( S = I \) and equation (4) is the usual definition of concentration parameter in Bound et al. (1995); Mariano (1973); Staiger and Stock (1997); Stock and Wright (2000) using population quantities, i.e. \( \Lambda_{S|S^C} \); for instance, \( C(S) \) corresponds exactly to the quantity \( \lambda' / \lambda K_2 \) on page 561 of Staiger and Stock (1997) for \( n = 1 \) and \( K_1 = 0 \). Without using population quantities, the function \( nC(S) \) roughly corresponds to the usual concentration parameter using the estimated version of \( \Lambda_{S|S^C} \). However, if only a subset of all instruments are relevant so that \( S \subset I \), then the concentration parameter represents the strength of instruments for that subset \( S \), adjusted for the exogenous variables in its complement \( S^C \). Regardless, like the usual concentration parameter, a high value of \( C(S) \) represents strong instruments in the set \( S \) while a low value of \( C(S) \) represents weak instruments.

### 2.3 The DWH Test

Consider the following hypotheses for detecting endogeneity in models (2) and (3),

\[
H_0 : \Sigma_{12} = 0, \quad H_1 : \Sigma_{12} \neq 0.
\]  

(5)

The DWH test tests for endogeneity as specified by the hypothesis in equation (5) by comparing two consistent estimators of \( \beta \) under the null hypothesis \( H_0 \) of no endogeneity, with
different efficiencies. Specifically, the DWH test statistic, denoted as \( Q_{DWH} \), is the quadratic difference between the OLS estimator of \( \beta \), formally \( \hat{\beta}_{OLS} = (D'P_{X⊥}D)^{-1}D'P_{X⊥}Y \), and the TSLS estimator of \( \beta \), formally \( \hat{\beta}_{TSLS} = (D'(P_{W} - P_{X})D)^{-1}D'(P_{W} - P_{X})Y \);

\[
Q_{DWH} = \frac{(\hat{\beta}_{TSLS} - \hat{\beta}_{OLS})^2}{\text{Var}(\hat{\beta}_{TSLS}) - \text{Var}(\hat{\beta}_{OLS})}. \tag{6}
\]

The terms \( \text{Var}(\hat{\beta}_{OLS}) \) and \( \text{Var}(\hat{\beta}_{TSLS}) \) are standard error estimates of the OLS and TSLS estimators, respectively, and have the following forms

\[
\text{Var}(\hat{\beta}_{OLS}) = (D'P_{X⊥}D)^{-1}\Sigma_{11}, \quad \text{Var}(\hat{\beta}_{TSLS}) = (D'(P_{W} - P_{X})D)^{-1}\Sigma_{11}. \tag{7}
\]

The \( \hat{\Sigma}_{11} \) in equation (7) is the estimate of \( \Sigma_{11} \) and can either be based on the OLS estimate of \( \Sigma \), i.e. \( \hat{\Sigma}_{11} = \|Y - D\hat{\beta}_{OLS} - X\hat{\phi}_{OLS}\|_2^2/n \), or the TSLS estimate of \( \Sigma \), i.e. \( \hat{\Sigma}_{11} = \|Y - D\hat{\beta}_{TSLS} - X\hat{\phi}_{TSLS}\|_2^2/n \).\(^5\) Under \( H_0 \), both OLS and TSLS estimators of the variance \( \Sigma_{11} \) are consistent. Also, under \( H_0 \), both OLS and TSLS estimators are typically consistent estimators of \( \beta \), but the OLS estimator is more efficient than the TSLS estimator.

The asymptotic null distribution of the DWH test in equation (6) is a Chi-squared distribution with one degree of freedom. With a known \( \Sigma_{11} \), the DWH test has an exact Chi-squared null distribution with one degree of freedom. Regardless, both null distributions imply that for each \( \alpha, 0 < \alpha < 1 \), we can reject the null hypothesis \( H_0 \) of no endogeneity for the alternative \( H_1 \) by using the decision rule,

\[
\text{Reject } H_0 \text{ if } Q_{DWH} \geq \chi^2_{\alpha}(1)
\]

and the Type I error of the DWH test will be asymptotically (or exactly if \( \Sigma_{11} \) is known) controlled at level \( \alpha \). Also, under the local alternative hypothesis,

\[
H_0 : \Sigma_{12} = 0, \quad H_2 : \Sigma_{12} = \frac{\Delta_{1}}{\sqrt{n}} \tag{8}
\]

\(^5\)To be precise, the OLS and TSLS estimates of \( \phi \) can be obtained as follows:
\( \hat{\phi}_{OLS} = (X'P_{D⊥}X)^{-1}X'P_{D⊥}Y \) and \( \hat{\phi}_{TSLS} = (X'P_{D⊥}X)^{-1}X'P_{D⊥}Y \) where \( D = P_{W}D \).
for some constant $\Delta_1 \neq 0$, the asymptotic power of the DWH test is

$$\omega \in H_2 : \lim_{n \to \infty} P(Q_{\text{DWH}} \geq \chi^2_0(1)) = G \left( \alpha, \frac{\Delta_1 \sqrt{C(I)}}{\sqrt{\left( C(I) + \frac{1}{p_z} \right) \Sigma_{11} \Sigma_{22}}} \right),$$

(9)

where $G(\alpha, \cdot)$ is defined in equation (1); see supplementary material Theorem 3 for a proof of (9). For textbook discussions on the DWH test, see Section 7.9 of Davidson and MacKinnon (1993) and Section 6.3.1 of Wooldridge (2010).

3 The DWH Test with Many Covariates

We now consider the behavior of the DWH test in the presence of many covariates and/or instruments. Formally, suppose the number of covariates and instruments are growing with sample size $n$, $p_x = p_x(n)$ and $p_z = p_z(n)$, so that $p = p_x + p_z$ and $n - p$ are increasing with respect to $n$. For this section only, we focus on the case where $p < n$ since the DWH test with OLS and TSLS estimators cannot be implemented when the sample size is smaller than the dimension of the model parameters; later sections, specifically Section 4 will consider endogeneity testing in $p \geq n$ settings. We assume a known $\Sigma_{11}$ for a cleaner technical exposition and to highlight the deficiencies of the DWH test that’s not specific to estimating $\Sigma_{11}$, but specific to the form of the DWH test, the quadratic differencing of estimators in equation (6). However, the known $\Sigma_{11}$ assumption can be replaced by a consistent estimate of $\Sigma_{11}$. Theorem 1 characterizes the asymptotic behavior of the DWH test under this setting.

**Theorem 1.** Suppose we have models (2) and (3) where $\Sigma_{11}$ is known and $W_i$ is a zero-mean multivariate Gaussian. If $\sqrt{C(I)} \gg \sqrt{\log(n - p_x)/(n - p_x)p_z}$, for each $\alpha$, $0 < \alpha < 1$, the asymptotic Type I error of the DWH test under $H_0$ is controlled at $\alpha$

$$\omega \in H_0 : \limsup_{n, p_x, p_z \to \infty} P \left( |Q_{\text{DWH}}| \geq z_{\alpha/2} \right) = \alpha.$$
Furthermore, the asymptotic power of the DWH test under $H_2$ is
\[
\omega \in H_2 : \lim_{n,p_x,p_z \to \infty} \left| P \left( Q_{\text{DWH}} \geq \chi^2_0(1) \right) - G \left( \alpha, \frac{C(I)\Delta_1\sqrt{1 - \frac{p}{n}}}{\sqrt{\left( C(I) + \frac{1}{n-p_x} \right) \left( C(I) + \frac{1}{p_z} \right) \Sigma_{11}\Sigma_{22}}} \right) \right| = 0.
\] (10)

Surprisingly, Theorem 1 states that the Type I error of the DWH test is actually controlled at the desired level $\alpha$ if one were to naively use it in the presence of many covariates and/or instruments. However, the power of the DWH test under the local alternative $H_2$ in equation (10) behaves differently in high dimensions than in low dimensions, as specified in equation (9). For example, if covariates and/or instruments are growing at $p/n \to 0$, equation (10) reduces to the usual power of the DWH test under low dimensional settings in equation (9). On the other hand, if covariates and/or instruments are growing at $p/n \to 1$, then the usual DWH test essentially has no power against any local alternative in $H_2$ since $G(\alpha, \cdot)$ in equation (10) equals $\alpha$ for any value of $\Delta_1$.

This phenomena suggests that in the “middle ground” where $p/n \to c$, $0 < c < 1$, the usual DWH test will likely suffer in terms of power. As a concrete example, if $p_x = n/2$ and $p_z = n/3$ so that $p/n = 5/6$, then $G(\alpha, \cdot)$ in equation (10) reduces to
\[
G \left( \alpha, \frac{C(I)\Delta_1}{\sqrt{2 \left( C(I) + \frac{2}{n} \right) \left( C(I) + \frac{1}{p_z} \right) \Sigma_{11}\Sigma_{22}}} \right) \approx G \left( \alpha, \frac{1}{\sqrt{6}} \frac{\sqrt{C(I)\Delta_1}}{\sqrt{\left( C(I) + \frac{1}{p_z} \right) \Sigma_{11}\Sigma_{22}}} \right)
\]
where the approximation sign is for $n$ sufficiently large enough so that $C(I) + 2/n \approx C(I)$. Under this setting, the power of the DWH test is smaller than the power in equation (9) in low dimension. Section 6 also shows this phenomena numerically.

Also, Theorem 1 provides some important guidelines for empiricists using the DWH test. First, Theorem 1 suggests that with modern cross-sectional data where the number of covariates may be very large, the DWH test should not be used to test endogeneity. Not only is the DWH test potentially incapable of detecting the presence of endogeneity under this scenario, but also an empiricist may be misled into a non-IV type of analysis, say the OLS or the Lasso, based on the result of the DWH test (Wooldridge, 2010). If the
empiricist used a more powerful endogeneity test under this setting, he or she would have correctly concluded that there is endogeneity and used an IV analysis. Second, as discussed in Section 1, if empirical works add many covariates to make an IV more plausibly valid, one pays a price in terms of the power of the specification test and additional sample size may be needed to get the desired level of power for detecting endogeneity.

Finally, we make two remarks about the regularity condition in Theorem 1. First, Theorem 1 controls the growth of the concentration parameter $C(I)$ to be faster than $\log(n - p_x)/(n - p_x)p_z$. This growth condition is satisfied under the many instrument asymptotics of Bekker (1994) and the many weak instrument asymptotics of Chao and Swanson (2005) where $C(I)$ converges to a constant as $p_z/n \to c$ for some constant $c$.

The weak instrument asymptotics of Staiger and Stock (1997) are not directly applicable to our growth condition on $C(I)$ because the asymptotics keeps $p_z$ and $p_x$ fixed. Second, we can replace the condition that $W_i$ is a zero-mean multivariate Gaussian in Theorem 1 by another condition used in high dimensional IV regression, for instance page 486 of Chernozhukov et al. (2015) where (i) the vector of instruments $Z_i$ is a linear model of $X_i$, i.e. $Z_i' = X_i' B + \bar{Z}_i'$, (ii) $\bar{Z}_i$ is independent of $X_i$, and (iii) $\bar{Z}_i$ is a multivariate normal distribution and the results in Theorem 1 will hold.

4 An Improved Endogeneity Test

Given that the DWH test for endogeneity may have low power when high dimensional variables are present, we present a simple and improved endogeneity test that has better power to detect endogeneity. In particular, our endogeneity test takes any popular estimator that is “well-behaved” for estimating reduced-form parameters (see Definition 2 for details) and applies a simple hard thresholding procedure to choose the most relevant instruments. We also stress that our endogeneity test is the first test capable of testing endogeneity if the number of parameters exceeds the sample size.
4.1 Well-Behaved Estimators

Consider the following reduced-form models

\[ Y_i = Z_i' \Gamma + X_i' \Psi + \xi_i, \]  
\[ D_i = Z_i' \gamma + X_i' \psi + \epsilon_i. \]

The terms \( \Gamma = \beta \gamma \) and \( \Psi = \phi + \beta \psi \) are the parameters for the reduced-form model (11) and \( \xi_i = \beta \epsilon_i + \delta_i \) is the reduced-form error term. The errors in the reduced-form models have the property that \( E(\xi_i|Z_i, X_i) = 0 \) and \( E(\epsilon_i|Z_i, X_i) = 0 \). Also, the covariance matrix of these error terms, denoted as \( \Theta \), have the following forms: \( \Theta_{11} = \text{Var}(\xi_i|Z_i, X_i) = \Sigma_{11} + 2\beta \Sigma_{12} + \beta^2 \Sigma_{22} \), \( \Theta_{22} = \text{Var}(\epsilon_i|Z_i, X_i) \), and \( \Theta_{12} = \text{Cov}(\xi_i, \epsilon_i|Z_i, X_i) = \Sigma_{12} + \beta \Sigma_{22} \).

As mentioned before, our improved endogeneity test does not require a specific estimator for the reduced-form parameters. Rather, any estimator that is well-behaved, as defined below, will be sufficient.

**Definition 2.** Consider estimators \( (\hat{\gamma}, \hat{\Gamma}, \hat{\Theta}_{11}, \hat{\Theta}_{22}, \hat{\Theta}_{12}) \) of the reduced-form parameters, \( (\gamma, \Gamma, \Theta_{11}, \Theta_{22}, \Theta_{12}) \) respectively, in equations (11) and (12). The estimators \( (\hat{\gamma}, \hat{\Gamma}, \hat{\Theta}_{11}, \hat{\Theta}_{22}, \hat{\Theta}_{12}) \) are well-behaved estimators if they satisfy two criterions

**W1** The reduced-form estimators of the coefficients \( \hat{\gamma} \) and \( \hat{\Gamma} \) satisfy

\[ \sqrt{n} \| (\hat{\gamma} - \gamma) - \frac{1}{n} \hat{V}' \epsilon \|_\infty = O_p \left( \frac{s \log p}{\sqrt{n}} \right), \quad \sqrt{n} \| (\hat{\Gamma} - \Gamma) - \frac{1}{n} \hat{V}' \xi \|_\infty = O_p \left( \frac{s \log p}{\sqrt{n}} \right). \]

for some matrix \( \hat{V} = (\hat{V}_1, \ldots, \hat{V}_{p_z}) \) which is only a function of \( W \) and satisfies

\[ \lim \inf \inf_{n \to \infty} \inf_{\omega \in \Omega} P \left( c \leq \min_{1 \leq j \leq p_z} \frac{\| \hat{V}_j \|_2}{\sqrt{n}} \leq \max_{1 \leq j \leq p_z} \frac{\| \hat{V}_j \|_2}{\sqrt{n}} \leq C, \epsilon \| \gamma \|_2 \leq \frac{1}{\sqrt{n}} \left\| \sum_{j \in S} \gamma_j \hat{V}_j \|_2 \right\| = 1 \]

for some constants \( c > 0 \) and \( C > 0 \).
The reduced-form estimators of the error variances, $\hat{\Theta}_{11}$, $\hat{\Theta}_{22}$, and $\hat{\Theta}_{12}$ satisfy
\[
\sqrt{n} \max \left\{ \left| \frac{1}{\sqrt{n}} \hat{\Theta}_{11} - \frac{1}{n} \xi' \xi \right|, \left| \frac{1}{\sqrt{n}} \hat{\Theta}_{12} - \frac{1}{n} \epsilon' \xi \right|, \left| \frac{1}{\sqrt{n}} \hat{\Theta}_{22} - \frac{1}{n} \epsilon' \epsilon \right| \right\} = O_p \left( \frac{s \log p}{\sqrt{n}} \right).
\] (15)

There are many estimators for the reduced-form parameters in the literature that are well-behaved. Some examples of well-behaved estimators are listed below.

1. (OLS): In settings where $p$ is fixed or $p$ is growing with $n$ at a rate $p/n \to 0$, the OLS estimators of the reduced-form parameters, i.e.
\[
(\hat{\Gamma}, \hat{\Psi}) = (W'W)^{-1}W'Y, (\hat{\gamma}, \hat{\psi}) = (W'W)^{-1}W'D.
\]
\[
\hat{\Theta}_{11} = \frac{n}{n} \left( Y - Z\hat{\Gamma} - X\hat{\Psi} \right)' \left( D - Z\hat{\gamma} - X\hat{\psi} \right)\frac{2}{n} \left( D - Z\hat{\gamma} - X\hat{\psi} \right)\frac{2}{n}
\]

trivially satisfy conditions for well-behaved estimators. Specifically, let $\hat{\nu}' = (\frac{1}{n}W'W)_{T}^{-1}W$.
Then equation (13) holds because $(\hat{\gamma} - \gamma)' - \hat{\nu}'(\hat{\nu}'(\hat{\nu}' - \nu)' = 0$ and $(\hat{\Gamma} - \Gamma)' - \hat{\nu}'(\hat{\nu}' - \nu)' = 0$. Also, equation (14) holds because, in probability, $n^{-1/2}\|\hat{\nu}'\|_2 \to \Lambda^{-1}_{jj}$ and $n^{-1/2}\hat{\nu}' \hat{\nu} \to \Lambda^{-1}_{zz}$, thus satisfying (W1). Also, (W2) holds because $\|\hat{\Gamma} - \Gamma\|_2^2 + \|\hat{\Psi} - \Psi\|_2^2 = O_p(n^{-1})$ and $\|\hat{\gamma} - \gamma\|_2^2 + \|\hat{\psi} - \psi\|_2^2 = O_p(n^{-1})$, which implies equation (15) going to zero at $n^{-1/2}$ rate.

2. (Debiased Lasso Estimators) In high dimensional settings where $p$ is growing with $n$ and often exceeds $n$, one of the most popular estimators for regression model parameters is the Lasso (Tibshirani, 1996). Unfortunately, the Lasso estimator, let alone many penalized estimators, do not satisfy the definition of a well-behaved estimator, specifically (W1), because penalized estimators are typically biased. Fortunately, recent works by Javanmard and Montanari (2014); van de Geer et al. (2014); Zhang and Zhang (2014) and Cai and Guo (2016) remedied this bias problem by doing a bias correction on the original penalized estimates.
More concretely, suppose we use the square root Lasso estimator by Belloni et al. (2011b),

$$\{\tilde{\Gamma}, \tilde{\Psi}\} = \arg\min_{\Gamma \in \mathbb{R}^{pz}, \Psi \in \mathbb{R}^{px}} \frac{1}{\sqrt{n}} \|Y - Z\Gamma - X\Psi\|_2 + \frac{\lambda_0}{\sqrt{n}} \left( \sum_{j=1}^{pz} \|Z_j\|_2 |\Gamma_j| + \sum_{j=1}^{px} \|X_j\|_2 |\Psi_j| \right)$$  \hspace{1cm} (16)

for the reduced-form model in equation (11) and

$$\{\tilde{\gamma}, \tilde{\psi}\} = \arg\min_{\Gamma \in \mathbb{R}^{pz}, \Psi \in \mathbb{R}^{px}} \frac{1}{\sqrt{n}} \|D - Z\tilde{\gamma} - X\tilde{\psi}\|_2 + \frac{\lambda_0}{\sqrt{n}} \left( \sum_{j=1}^{pz} \|Z_j\|_2 |\gamma_j| + \sum_{j=1}^{px} \|X_j\|_2 |\psi_j| \right)$$  \hspace{1cm} (17)

for the reduced-form model in equation (12). The term $\lambda_0$ in both estimation problems (16) and (17) represents the penalty term in the square root Lasso estimator and typically, the penalty is set at $\lambda_0 = \sqrt{a_0 \log p/n}$ for some constant $a_0$ slightly greater than 2, say 2.01 or 2.05. To transform the above penalized estimators in equations (16) and (17) into well-behaved estimators, we can follow Javanmard and Montanari (2014) to debias the penalized estimators. Specifically, we solve $p_z$ optimization problems where the solution to each $p_z$ optimization problem, denoted as $\hat{u}[j] \in \mathbb{R}^p$, $j = 1, \ldots, p_z$, is

$$\hat{u}[j] = \arg\min_{u \in \mathbb{R}^p} \frac{1}{n} \|Wu\|^2_2 \quad \text{s.t.} \quad \frac{1}{n} W'Wu - I_j \|_{\infty} \leq \lambda_n.$$  

Typically, the tuning parameter $\lambda_n$ is chosen to be $12M_1^2 \sqrt{\log p/n}$ where $M_1$ defined as the largest eigenvalue of $\Gamma$. Define $\hat{V}_j = W\hat{u}[j]$ and $\hat{V} = (\hat{V}_1, \ldots, \hat{V}_{p_z})$. Then, we can transform the penalized estimators in (16) and (17) into the debiased, well-behaved estimators, $\hat{\Gamma}$ and $\hat{\gamma}$,

$$\hat{\Gamma} = \tilde{\Gamma} + \frac{1}{n} \hat{V}' \left( Y - Z\tilde{\Gamma} - X\tilde{\Psi} \right), \quad \hat{\gamma} = \tilde{\gamma} + \frac{1}{n} \hat{V}' \left( D - Z\tilde{\gamma} - X\tilde{\psi} \right).$$  \hspace{1cm} (18)

See Guo et al. (2016) for details showing $\hat{\Gamma}$ and $\hat{\gamma}$ satisfy (W1). As for the error variances, following Belloni et al. (2011b), Sun and Zhang (2012) and Ren et al. (2013), we estimate the covariance terms $\Theta_{11}, \Theta_{22}, \Theta_{12}$ by using the estimates from
equations (16) and (17)

\[ \hat{\Theta}_{11} = \frac{\|Y - Z\tilde{\Gamma} - X\tilde{\Psi}\|_2^2}{n}, \hat{\Theta}_{22} = \frac{\|D - Z\tilde{\gamma} - X\tilde{\psi}\|_2^2}{n} \]

\[ \hat{\Theta}_{12} = \frac{(Y - Z\tilde{\Gamma} - X\tilde{\Psi})'(D - Z\tilde{\gamma} - X\tilde{\psi})}{n} \]

Lemma 3 of Guo et al. (2016) shows that the above estimators of \(\hat{\Theta}_{11}, \hat{\Theta}_{22}\) and \(\hat{\Theta}_{12}\) in equation (19) satisfy (W2). In summary, the debiased Lasso estimators in equation (18) and the variance estimators in equation (19) are well-behaved estimators.

3. (One-Step and Orthogonal Estimating Equations Estimators) Recently, Chernozhukov et al. (2015) proposed the one-step estimator of the reduced-form coefficients, i.e.

\[ \hat{\Gamma} = \tilde{\Gamma} + \frac{1}{n}\hat{\Lambda}^{-1}_Z, W^T (Y - Z\tilde{\Gamma} - X\tilde{\Psi}), \hat{\gamma} = \tilde{\gamma} + \frac{1}{n}\hat{\Lambda}^{-1}_Z, W^T (D - Z\tilde{\gamma} - X\tilde{\psi}) \]

where \(\tilde{\Gamma}\) and \(\tilde{\gamma}\) and \(\hat{\Lambda}^{-1}\) are initial estimators of \(\Gamma, \gamma\) and \(\Lambda^{-1}\), respectively. The initial estimators must satisfy conditions (18) and (20) of Chernozhukov et al. (2015) and many popular estimators like the Lasso or the square root Lasso satisfy these two conditions. Then, the arguments in Theorem 2.1 of van de Geer et al. (2014) show that the one-step estimator of Chernozhukov et al. (2015) satisfies (W1). Relatedly, Chernozhukov et al. (2015) proposed estimators for the reduced-form coefficients based on orthogonal estimating equations and in Proposition 4 of Chernozhukov et al. (2015), the authors showed that the orthogonal estimating equations estimator is asymptotically equivalent to their one-step estimator.

For variance estimation, one can use the variance estimator in Belloni et al. (2011b), which reduces to the estimators in equation (19), satisfying (W2).

In short, the first part of our endogeneity test requires any estimator that is well-behaved and, as illustrated above, many estimators, such as the OLS and variants of the Lasso, satisfy the criteria for a well-behaved estimator.
4.2 Estimating Relevant Instruments via Hard Thresholding

Once we have well-behaved estimators ($\hat{\gamma}$, $\hat{\Gamma}$, $\hat{\Theta}_{11}$, $\hat{\Theta}_{22}$, $\hat{\Theta}_{12}$) satisfying Definition 2, the next step in our endogeneity test is finding IVs that are relevant, that is the set $S$ in Definition 1 comprised of $\gamma_j \neq 0$. We do this by hard thresholding the estimate $\hat{\gamma}$ of $\gamma$ by the dimension and the noise of $\hat{\gamma}$.

$$\hat{S} = \left\{ j : |\hat{\gamma}_j| \geq \frac{\sqrt{\hat{\Theta}_{22}}}{\sqrt{n}} \left( \frac{a_0 \log \max\{p_z, n\}}{n} \right)^{1/2} \right\}. \tag{20}$$

The set $\hat{S}$ is an estimate of $S$ and $a_0$ is some constant greater than 2; from our experience and like many Lasso problems, $a_0 = 2.01$ or $a_0 = 2.05$ work well in practice. The threshold in (20) is based on the noise level of $\hat{\gamma}_j$ in equation (13) (represented by the term $n^{-1/2} \sqrt{\hat{\Theta}_{22}} ||\hat{V}_j||_2$), adjusted by the dimensionality of the instrument size (represented by the term $\sqrt{a_0 \log \max\{p_z, n\}}$).

Then, using the estimated set $\hat{S}$ of relevant IVs, we obtain our estimates of $\Sigma_{12}$, $\Sigma_{11}$, and $\beta$

$$\hat{\Sigma}_{12} = \hat{\Theta}_{12} - \hat{\beta} \hat{\Theta}_{22}, \quad \hat{\Sigma}_{11} = \hat{\Theta}_{11} + \hat{\beta}^2 \hat{\Theta}_{22} - 2 \hat{\beta} \hat{\Theta}_{12}, \quad \hat{\beta} = \frac{\sum_{j \in \hat{S}} \hat{\gamma}_j \hat{\Gamma}_j}{\sum_{j \in \hat{S}} \hat{\gamma}_j^2}. \tag{21}$$

Equation (21) provides us with the ingredients to construct our new test for endogeneity, which we denote as $Q$

$$Q = \frac{n \hat{\Sigma}_{12}}{\sqrt{\text{Var}(\hat{\Sigma}_{12})}}, \quad \text{Var}(\hat{\Sigma}_{12}) = \hat{\Theta}_{22}^2 \text{Var}_1 + \text{Var}_2 \tag{22}$$

where $\text{Var}_1 = \hat{\Sigma}_{11} \left\| \sum_{j \in \hat{S}} \hat{\gamma}_j \hat{V}_j \right\|_2^2 / \left( \sum_{j \in \hat{S}} \hat{\gamma}_j^2 \right)^2$ and $\text{Var}_2 = \hat{\Theta}_{11} \hat{\Theta}_{22} + \hat{\Theta}_{12}^2 + 2 \hat{\beta}^2 \hat{\Theta}_{22}^2 - 4 \hat{\beta} \hat{\Theta}_{12} \hat{\Theta}_{22}$. Here, Var$_1$ is the variance associated with estimating $\beta$ and Var$_2$ is the variance associated with estimating $\Theta$.

Note that the key difference between the original DWH test in equation (6) and our endogeneity test in equation (22) is that our endogeneity test directly estimates and tests the endogeneity parameter $\Sigma_{12}$ while the original DWH test implicitly tests for the endogeneity...
parameter by checking the quadratic distance between the OLS and TSLS estimators under the null hypothesis. As we will show in Section 4.3, our endogeneity test in this form where we directly estimate $\Sigma_{12}$ will have superior power in high dimensions compared to the DWH test.

### 4.3 Properties of the New Endogeneity Test

We study the properties of our new test in high dimensional settings where $p$ is a function of $n$ and is allowed to be larger than $n$; note that this is a generalization of the setting discussed in Section 3 where $p < n$ because the DWH test is not feasible when $p \geq n$. Theorem 1 showed that the DWH test, while it controls Type I error at the desired level, may have low power, especially when the ratio of $p/n$ is close to 1. Theorem 2 shows that our new test $Q$ remedies this deficiency of the DWH test by having proper Type I error control and exhibiting better power than the DWH test.

**Theorem 2.** Suppose we have models (2) and (3) and we use a well-behaved estimator in our test statistic $Q$. If $\sqrt{\mathcal{C}(S)} \gg s_{z1} \log p/\sqrt{n|V|}$, and $\sqrt{s_{z1}} \log p/\sqrt{n} \to 0$, then for any $\alpha$, $0 < \alpha < 1$, the asymptotic Type I error of $Q$ under $H_0$ is controlled at $\alpha$

$$\omega \in H_0 : \lim_{n,p,x,z \to \infty} P(\{|Q| \geq z_{\alpha/2}\} = \alpha,$$

and the asymptotic power of $Q$ under $H_2$ is

$$\limsup_{n,p,z,x \to \infty} \left| P(\{|Q| \geq z_{\alpha/2}\} - E \left( G\left( \alpha, \frac{\Delta_1}{\sqrt{\Theta_2 Var_1 + Var_2}} \right) \right) \right| = 0,$$

with $G(\alpha, \cdot)$ defined in (1), $Var_1 = \Sigma_{11} \left\| \sum_{j \in S} \gamma_j \tilde{V}_j / \sqrt{n} \right\|_2^2 / \left( \sum_{j \in S} \gamma_j^2 \right)^2$ and $Var_2 = \Theta_{11} \Theta_{22} + \Theta_{12}^2 + 2\beta^2 \Theta_{22}^2 - 4\beta \Theta_{12} \Theta_{22}$.

In contrast to equation (10) that described the power of the usual DWH test in high dimensions, the term $\sqrt{1 - p/n}$ is absent in the power of our new endogeneity test in equation (24). Specifically, under the local alternative $H_2$, our power is only affected by $\Delta_1$ while the power of the DWH test is affected by $\Delta_1 \sqrt{1 - p/n}$. Consequently, the power of
our test \( Q \) do not suffer from the growing dimensionality of \( p \). For example, in the extreme case when \( p/n \to 1 \) and \( C(S) \) is a constant, the power of the usual DWH test will be \( \alpha \) while the power of our test \( Q \) will always be greater than \( \alpha \). As further validation, the simulation in Section 6 numerically illustrates the discrepancies between the power of the two tests. Finally, we stress that in the case \( p > n \), our test still has proper size and non-trivial power while the DWH test is not feasible in this setting.

With respect to the regularity conditions in Theorem 2, like Theorem 1, Theorem 2 controls the growth of the concentration parameter \( C(S) \) to be faster than \( s_{z1} \log p/\sqrt{n|S|} \), with a minor discrepancy in the growth rate due to the differences between the set of relevant IVs, \( S \), and the set of candidate IVs, \( I \). But, similar to Theorem 1, this growth condition is satisfied under the many instrument asymptotics of Bekker (1994) and the many weak instrument asymptotics of Chao and Swanson (2005). Also, note that unlike the negative result in Theorem 1, the “positive” result in Theorem 2 is more general in that we do not explicitly require \( W \) to be Gaussian. Instead, we only need the conditions of well-behaved estimators to be satisfied. Finally, we remark that the expectation inside equation (24) is respect to \( W \) and \( \hat{V} \) is a function of \( W \).

5 An Extension: Endogeneity Test in High Dimensions with Possibly Invalid IVs

As discussed in Section 1, one of the motivation for having high dimensional covariates in empirical IV work is to avoid invalid instruments. While adding more covariates can potentially make instruments more plausibly valid, as demonstrated in Section 3, there is a price to pay with respect to the power of the DWH test. More importantly, even after conditioning on many covariates, some IVs may still be invalid and subsequent analysis, including the DWH test, assuming that all the IVs are valid after conditioning can be seriously misleading.

Inspired by these concerns, there has been a recent literature in estimation and inference of structural parameters in IV regression when invalid instruments are present.
2016; Kang et al., 2016; Kolesár et al., 2015). In this line of work, the invalid instruments are represented as direct effects between the instruments and the outcome in equation (2), i.e.

\[ Y_i = D_i \beta + Z_i' \pi + X_i' \phi + \delta_i, \quad E(\delta_i \mid Z_i, X_i) = 0 \]  

If \( \pi = 0 \) in model (25), the model (25) reduces to the usual instrumental variables regression model in equation (2) with one endogenous variable, \( p_x \) exogenous covariates, and \( p_z \) instruments, all of which are assumed to be valid. On the other hand, if \( \pi \neq 0 \) and the support of \( \pi \) is unknown a priori, the instruments may have a direct effect on the outcome, thereby violating the exclusion restriction (Angrist et al., 1996; Imbens and Angrist, 1994), without knowing, a priori, which are invalid and valid (Conley et al., 2012; Kang et al., 2016; Murray, 2006). In short, the support of \( \pi \) allows us to distinguish a valid instrument, i.e. \( \pi_j = 0 \) from an invalid one, i.e. \( \pi_j \neq 0 \).

Despite the presence of invalid IVs, our new endogeneity test can handle this case. Specifically, following Section 3.3 of Guo et al. (2016), we can estimate \( \pi \) in the model (25) by taking each IV \( j \) that are estimated to be relevant, i.e. \( j \in \hat{S} \), and constructing a candidate estimate of \( \pi \) by using this IV, i.e. \( \tilde{\pi}^{[j]} = \hat{\Gamma} - \hat{\Gamma}_j / \hat{\gamma}_j \hat{\gamma} \). We also define \( \hat{\Sigma}^{[j]}_{11} \) to be the pilot estimate of \( \Sigma_{11} \), i.e. \( \hat{\Sigma}^{[j]}_{11} = \hat{\Theta}_{11} + (\hat{\beta}^{[j]} - 2) \hat{\Theta}_{22} - 2 \hat{\beta}^{[j]} \hat{\Theta}_{12} \). Then, for each \( \tilde{\pi}^{[j]} \) in \( j \in \hat{S} \), we threshold each element of \( \tilde{\pi}^{[j]} \) to create the thresholded estimate \( \hat{\pi}^{[j]} \),

\[
\hat{\pi}^{[j]}_k = \tilde{\pi}^{[j]}_k 1 \left( k \in \hat{S} \cap |\tilde{\pi}^{[j]}_k| \geq a_0 \sqrt{\hat{\Sigma}^{[j]}_{11}} \frac{\| \hat{V}_k - \frac{\hat{\gamma}_j}{\hat{\gamma}} \hat{V}_{j} \|_2}{\sqrt{n}} \sqrt{\log \max \{ p_x, n \} / n} \right)
\]

for all \( 1 \leq k \leq p_x \). Among the \( |\hat{S}| \) candidate estimates of \( \pi \), we choose \( \hat{\pi}^{[j]} \) with the most valid instruments, i.e. we choose \( j^* \in \hat{S} \) where \( j^* = \text{argmin} \| \hat{\pi}^{[j]} \|_0 \). Subsequently, we estimate \( \beta, \Sigma_{11} \) and \( \Sigma_{12} \) as in equation (21) except we replace \( \hat{S} \) with the set \( \hat{V} = \)
For additional details, see Section 3.3 of Guo et al. (2016). Also, the theoretical details of this modification with respect to Type I error and power are presented in Section 2.2 of the supplementary materials. In short, this extension still controls the Type I error rate and has non-negligible power under high dimensions with possibly invalid instruments.

6 Simulation

6.1 Setup

We conduct a simulation study to investigate the performance of our new endogeneity test and the DWH test in both cases low and high dimensional settings. For the low dimensional case, we generate data from models (2) and (3) in Section 2.2 with $p_z = 9$ instruments and $p_x = 5$ covariates. The sample size $n$ is set at 1000. The vector $W_i.$ is a multivariate normal with mean zero and covariance $\Lambda_{ij} = 0.5^{|i-j|}$ for $1 \leq i,j \leq p$. The parameters of the models are: $\beta = 1$, $\phi = (0.6, 0.7, 0.8, 0.9, 1.0) \in \mathbb{R}^5$ and $\psi = (1.1, 1.2, 1.3, 1.4, 1.5) \in \mathbb{R}^5$.

For the high dimensional case, we use the same models as the low dimensional case except $p_z = 100$, $p_x = 150$, $\phi = (0.6, 0.7, 0.8, \ldots, 1.5, 0, 0, \ldots, 0) \in \mathbb{R}^{p_x}$ so that $s_{x1} = 10$, and $\psi = (1.1, 1.2, 1.3, \ldots, 2.0, 0, 0, \ldots, 0) \in \mathbb{R}^{p_x}$ so that $s_{x2} = 10$. For sample size, we let $n = (200, 300)$. For both high dimensional and low dimensional cases, the relevant instruments are $S = \{1, \ldots, 7\}$. Variance of the error terms are set to $\text{Var}(\delta_i) = \text{Var}(\epsilon_i) = 1.5$.

The parameters we vary in the simulation study are: the endogeneity level via $\text{Cov}(\delta_i, \epsilon_i)$, and IV strength via $\gamma$. For the endogeneity level, we set $\text{Cov}(\delta_i, \epsilon_i) = 1.5 \rho$, where $\rho$ is varied and captures the level of endogeneity; a larger value of $|\rho|$ indicates a stronger correlation between the endogenous variable $D_i$ and the error term $\delta_i$. For IV strength, we set $\gamma_S = K (1, 1, 1, 1, 1, \rho_1)$ and $\gamma_{SC} = 0$, where $K$ is varied as a function of the concentration parameter (see below) and $\rho_1$ is varied based on the vector $(0, 0.1, 0.2)$ across simulations. Note that the value $K$ controls the global strength of instruments, with higher $|K|$ indicating strong instruments in a global sense. Also, the value $\rho_1$ controls the relative individual strength of instruments, specifically between the first four instruments in $S$ and the fifth...
instrument. For example, $\rho_1 = 0.2$ implies that the fifth IV’s individual strength is only 20% of the other four valid instruments, i.e IVs 1 to 4. Note that varying $\rho_1$ essentially stress-tests the thresholding step in our endogeneity test to numerically verify whether our testing procedure can handle relevant IVs with very small magnitudes of $\gamma$.

We specify $K$ as follows. Suppose we have a set of simulation parameters $S, \rho_1, \Lambda$ and $\Sigma_{22}$. For each value of $100 \cdot C(S)$, which simplifies to $100 \cdot C(S) = 100 \cdot K^2 \| \Lambda^{1/2}_{SI|SC}(1, 1, 1, 1, 1, \rho_1) \|_2^2 / (7 \cdot 1.5)$, we find $K$ that satisfies it. We vary $100 \cdot C(S)$ from 25 to 150, specifying $K$ for each value of $100 \cdot C(S)$.

We repeat data generation from each simulation setting specified above 500 times. For each simulation setting, we compare the power of our testing procedure to the DWH test and the oracle DWH test where an oracle knows the support of the parameter vectors $\phi, \psi$ and $\gamma$.

### 6.2 Results

First, Figure 1 considers the power of three comparators, our testing procedure $Q$, the regular DWH test and the oracle DWH test, under the low dimensional setting where $n = 1000, p_x = 9$, and $p_z = 5$. Columns “Strong”, “WeakIV1”, and “WeakIV2” represent different values of $\rho_1$, specifically $\rho_1 = 0$, $\rho_1 = 0.1$, and $\rho_1 = 0.2$ respectively. Each row in the figure represents different values of global IV strength measured by $nC(V)$, which as discussed before, corresponds roughly to the expected partial F-statistic assessing IV strength. The x-axis represents different values of endogeneity $\rho$ and the y-axis is the empirical proportion of rejecting the null hypothesis $H_0$ over 500 simulations and is an approximation to the test’s power. As expected, in low dimensional settings, the empirical power curves of the proposed test, the regular DWH test and the oracle DWH are identical.

Second, Figure 2 considers the power of our test $Q$, the regular DWH test, and the oracle DWH test in the high dimensional setting with $n = 300, p_x = 150$, and $p_z = 100$. All three tests have proper size, even the DWH test, which confirms our result in Theorem 1. However, as predicted by Theorem 1, the regular DWH test suffers from low power, especially if the degree of endogeneity is around 0.25 where the gap between the regular
Figure 1: Power of endogeneity tests when $n = 1000$, $p_x = 5$ and $p_z = 9$. The $x$-axis represents the endogeneity $\rho$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity, where the solid line, the dashed line and the dotted line represent the proposed test ($Q$), the regular DWH test and the oracle DWH test, respectively. The columns represent the individual IV strengths, with column names “Weak IV1”, “Weak IV2”, and “Strong IV” denoting the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent the overall strength of the instruments, as measured by $100 \cdot C(S)$. 

$\text{power.Proposed}$
DWH test and the oracle DWH test is the greatest across most of the simulation settings. In fact, even if the global strength of the IV increases, the DWH test still has low power. In contrast, our test $Q$ has uniformly better power than the regular DWH test across all degrees of endogeneity and across all simulation settings in the plot. Our test also achieves near-oracle performance as the global strength of the instrument grows. In addition, even if some of the relevant instruments are weak in the sense that their $\gamma$ values are close to zero (i.e. columns WeakIV1 and WeakIV2), our test does not suffer with respect to Type I error or power.

Finally, Figure 3 considers the power of our test $Q$ and the oracle DWH test in the high dimensional setting with $n = 200, p_x = 150$, and $p_z = 100$ so that $n < p$; we do not use the regular DWH test since the regular DWH is infeasible when $n < p$. As predicted from Theorem 2, our test $Q$ can handle the $n < p$ setting with the proper size and decent power. In particular, as the global strength of the IV increases, our test $Q$ converges to the oracle DWH test that has more information about the support of the parameter vectors. Also, as before, our test is generally immune to some relevant IVs having very weak strength, as measured by the variation in $\rho_1$ across the columns of Figure 3.

All the simulation results indicate that irrespective of the individual strength of the relevant IVs, our endogeneity test controls Type I error and is a much better alternative to the regular DWH test in high dimensional settings, with near-optimal performance with respect to the oracle. Our test is also capable of handling the regime $n < p$.

7 Conclusion

In this paper, we showed that the popular DWH test, while being able to control Type I error, can have relatively low power in high dimensional settings. We propose a simple and improved endogeneity test to remedy the low power of the DWH test by modifying popular reduced-form parameters with a simple thresholding step. We also show that this simple modification leads to drastically better power than the DWH test in high dimensional settings.
Figure 2: Power of endogeneity tests when $n = 300$, $p_x = 150$ and $p_z = 100$. The $x$-axis represents the endogeneity $\rho$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity, where the solid line, the dashed line and the dotted line represent the proposed test ($Q$), the regular DWH test and the oracle DWH test, respectively. The columns represent the individual IV strengths, with column names “Weak IV1”, “Weak IV2”, and “Strong IV” denoting the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent the overall strength of the instruments, as measured by $100 \cdot C(S)$. 

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Figure 3: Power of endogeneity tests when $n = 200$, $p_x = 150$ and $p_z = 100$. The $x$-axis represents the endogeneity $\rho$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity, where the solid line, the dashed line and the dotted line represent the proposed test ($Q$), the regular DWH test and the oracle DWH test, respectively. The columns represent the individual IV strengths, with column names “Weak IV1”, “Weak IV2”, and “Strong IV” denoting the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent the overall strength of the instruments, as measured by $100 \cdot C(S)$. 

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For empirical work, the results in the paper suggest that one should be cautious in interpreting high p-values produced by the DWH test in IV regression settings when many covariates and/or instruments are present. In particular, as Section 3 showed, in modern data settings with potentially large number of covariates and/or instruments, the DWH test may declare that there is no endogeneity in the structural model, even if endogeneity is truly present. Our proposed test, which is a simple modification of the popular estimators for reduced-forms parameters, does not suffer from this problem, as it achieves near-oracle performance to detect endogeneity, and can even handle general settings where invalid IVs are present in high dimension and when \( n < p \).

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**References**


Supplement to “Testing Endogeneity with High Dimensional Covariates”

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Abstract

This note summarizes the supplementary materials to the paper “Testing Endogeneity with High Dimensional Covariates”. In Section 1, we show that the DWH test fails in the presence of Invalid IVs. In Section 2, we discuss both method and theory for endogeneity test in high dimensions with invalid IVs. In Section 3, we present extended simulation studies that further examine the power and sensitivity of our test to regularity assumptions. In Section 4, we present technical proofs for Theorems 1, 2, 3, 4 and 5 and the proofs of technical lemmas.

1 Failure of the DWH Test in the Presence of Invalid IVs

While the DWH test performs as expected when all the instruments are valid, in practice, some instruments may be invalid and consequently, the DWH test can be a highly misleading assessment of the hypotheses (5). In Theorem 3, we show that the Type I error of the DWH
test can be greater than the nominal level for a wide range of IV configurations in which some IVs are invalid; we assume a known $\Sigma_{11}$ in Theorem 1 for a cleaner technical exposition and to highlight the impact that invalid IVs have on the size and power of the DWH test, but the known $\Sigma_{11}$ can be replaced by a consistent estimate of $\Sigma_{11}$. We also show that the power of the DWH test under the local alternative $H_2$ in equation (8) can be shifted.

**Theorem 3.** Suppose we have models (2) and (3) with a known $\Sigma_{11}$. If $\pi = \Delta_2/n^k$ where $\Delta_2$ is a fixed constant and $0 \leq k < \infty$, then for any $\alpha$, $0 < \alpha < 1$, we have the following asymptotic phase-transition behaviors of the DWH test for different values of $k$.

a. $0 \leq k < 1/2$: The asymptotic Type I error of the DWH test under $H_0$ is 1, i.e.

$$\omega \in H_0 : \lim_{n \to \infty} \mathbb{P}(Q_{DWH} \geq \chi_\alpha^2(1)) = 1$$

(26)

and the asymptotic power of the DWH test under $H_2$ is 1.

b. $k = 1/2$: The asymptotic Type I error of the DWH test under $H_0$ is

$$\omega \in H_0 : \lim_{n \to \infty} \mathbb{P}(Q_{DWH} \geq \chi_\alpha^2(1)) = G\left(\alpha, \frac{\frac{1}{p_z} \gamma' \Lambda_{I|I'} \Delta_2}{\sqrt{C(I) \left(C(I) + \frac{1}{p_z}\right) \Sigma_{11} \Sigma_{22}}}\right) \geq \alpha,$$

(27)

and the asymptotic power of the DWH test under $H_2$ is

$$\omega \in H_2 : \lim_{n \to \infty} \mathbb{P}(Q_{DWH} \geq \chi_\alpha^2(1)) = G\left(\alpha, \frac{\frac{1}{p_z} \gamma' \Lambda_{I|I'} \Delta_2}{\sqrt{C(I) \left(C(I) + \frac{1}{p_z}\right) \Sigma_{11} \Sigma_{22}}} + \frac{\Delta_1 \sqrt{C(I)}}{\sqrt{\left(C(I) + \frac{1}{p_z}\right) \Sigma_{11} \Sigma_{22}}}, \right).$$

(28)

where $G(\alpha, \cdot)$ is defined in (1).

c. $1/2 < k < \infty$: The asymptotic Type I error of the DWH test is $\alpha$, i.e.

$$\omega \in H_0 : \lim_{n \to \infty} \mathbb{P}(Q_{DWH} \geq \chi_\alpha^2(1)) = \alpha$$

(29)
and the asymptotic power of the DWH test under $H_2$ is equivalent to equation (9).

Theorem 3 presents the asymptotic behavior of the DWH test under a wide range of behaviors for the invalid IVs as represented by $\pi$. For example, when the instruments are invalid in the sense that their deviation from valid IVs (i.e. $\pi = 0$) to invalid IVs (i.e. $\pi \neq 0$) is at rates slower than $n^{-1/2}$, say $\pi = \Delta_2 n^{-1/4}$ or $\pi = \Delta_2$, equation (26) states that the DWH will always have Type I error and power that reach 1. In other words, if some IVs, or even a single IV, are moderately (or strongly) invalid in the sense that they have moderate (or strong) direct effects on the outcome above the usual noise level of the model error terms at $n^{-1/2}$, then the DWH test will always reject the null hypothesis of no endogeneity even if there is truly no endogeneity present; essentially, the DWH test behaves equivalently to a test that never looks at the data and always rejects the null.

Next, suppose the instruments are invalid in the sense that their deviation from valid IVs to invalid IVs are exactly at $n^{-1/2}$ rate, also referred to as the Pitman drift. This is the phase-transition point of the DWH test’s Type I error as the error moves from 1 in equation (26) to $\alpha$ in equation (29). Under this type of invalidity, equation (27) shows that the Type I error of the DWH test depends on some factors, most prominently the factor $\gamma' \Lambda_{I|\bar{I}} \Delta_2$. The factor $\gamma' \Lambda_{I|\bar{I}} \Delta_2$ has been discussed in the literature, most recently by Kolesár et al. (2015) within the context of invalid IVs. Specifically, Kolesár et al. (2015) studied the case where $\Delta_2 \neq 0$ so that there are invalid IVs, but $\gamma' \Lambda_{I|\bar{I}} \Delta_2 = 0$, which essentially amounted to saying that the IVs’ effect on the endogenous variable $D$ via $\gamma$ is orthogonal to their direct effects on the outcome via $\Delta_2$; see Assumption 5 of Section 3 in Kolesár et al. (2015) for details. Under their scenario, if $\gamma' \Lambda_{I|\bar{I}} \Delta_2 = 0$, then the DWH test will have the desired size $\alpha$. However, if $\gamma' \Lambda_{I|\bar{I}} \Delta_2$ is not exactly zero, which will most likely be the case in practice, then the Type I error of the DWH test will always be larger than $\alpha$ and we can compute the exact deviation from $\alpha$ by using equation (27). Also, equation (28) computes the power under $H_2$ in the $n^{-1/2}$ setting, which again depends on

\footnote{Fisher (1967) and Newey (1985) have used this type of $n^{-1/2}$ asymptotic argument to study misspecified econometrics models, specifically Section 2, equation (2.3) of Fisher (1967) and Section 2, Assumption 2 of Newey (1985). More recently, Hahn and Hausman (2005) and Berkowitz et al. (2012) used the $n^{-1/2}$ asymptotic framework in their respective works to study plausibly exogenous variables.}
the magnitude and direction of $\gamma'\Lambda_I|I^c\Delta_2$. For example, if there is only one instrument and that instrument has average negative effects on both $D$ and $Y$, the overall effect on the power curve will be a positive shift away from the case of valid IVs (i.e. $\pi = 0$). Regardless, under the $n^{-1/2}$ invalid IV regime, the DWH test will always have size that is at least as large as $\alpha$ if invalid IVs are present.

Theorem 3 also shows that instruments’ strength, as measured by the population concentration parameter $C(I)$ in equation (4), impacts the Type I error rate of the DWH test when the IVs are invalid at the $n^{-1/2}$ rate. Specifically, if $\pi = \Delta_2n^{-1/2}$ and the instruments are strong so that the concentration parameter $C(I)$ is large, then the deviation from $\alpha$ will be relatively minor even if $\gamma'\Lambda_I|I^c\Delta_2 \neq 0$. This phenomena has been mentioned in previous work, most notably Bound et al. (1995) and Angrist et al. (1996) where strong instruments can lessen the undesirable effects caused by invalid IVs.

Finally, if the instruments are invalid in the sense that their deviation from $\pi = 0$ is faster than $n^{-1/2}$, say $\pi = \Delta n^{-1}$, then equation (29) shows that the DWH test maintains its desired size. To put this invalid IV regime in context, if the instruments are invalid at $n^{-k}$ where $k > 1/2$, the convergence toward $\pi = 0$ is faster than the usual convergence rate of a sample mean from an i.i.d. sample towards a population mean. Also, this type of deviation is equivalent to saying that the invalid IVs are very weakly invalid and essentially act as if they are valid because the IVs are below the noise level of the model error terms at $n^{-1/2}$. Consequently, the DWH test is not impacted by these type of IVs with respect to size and power.

The overall implication of Theorem 3 is that whenever there is a concern for instrument validity, the results of the DWH test in practice should be scrutinized, especially when the DWH test produces low $p$-values. In particular, our theorem shows that the DWH test will only have correct size, (i) when the invalid IVs essentially behave as valid IVs asymptotically so that $\pi$’s rate toward zero is faster than usual mean convergence or (ii) when the IVs’ effects on the endogenous variables are completely orthogonal to each other. In all other settings, the Type I error of the DWH test will often be larger than $\alpha$ and consequently, the DWH test will tend to over-reject the null more frequently than it should, even if a single
invalid IV is present. In fact, the low p-value of the DWH test may mislead empiricists about the true presence of endogeneity; the endogenous variable may actually be exogenous and the low p-value may be entirely an artifact due to invalid IVs.

2 Endogeneity Test in High Dimensions with Invalid IVs

2.1 Method

Despite the presence of invalid IVs, our new endogeneity test can handle this case by using an additional thresholding procedure outlined in Section 3.3 of Guo et al. (2016a) to estimate π in the model (25). Specifically, we take each IV $j$ that are estimated to be relevant, i.e. $j \in \hat{S}$, and we define $\hat{\beta}^{[j]}$ to be a “pilot” estimate of $\pi$ by using this IV and dividing the reduced-form parameter estimates, i.e. $\hat{\pi}^{[j]} = \hat{\Gamma} - \hat{\beta}^{[j]} \hat{\gamma}$ where $\hat{\beta}^{[j]} = \hat{\Gamma}_j / \hat{\gamma}_j$. We also define $\hat{\pi}^{[j]}$ to be a pilot estimate of $\pi$ using this $j$th instrument’s estimate of $\beta$, i.e. $\hat{\pi}^{[j]} = \hat{\Gamma} - \hat{\beta}^{[j]} \hat{\gamma}$, and $\hat{\Sigma}_{11}^{[j]}$ to be the pilot estimate of $\Sigma_{11}$, i.e. $\hat{\Sigma}_{11}^{[j]} = \hat{\Theta}_{11} + (\hat{\beta}^{[j]})^2 \hat{\Theta}_{22} - 2 \hat{\beta}^{[j]} \hat{\Theta}_{12}$. Then, for each $\hat{\pi}^{[j]}$ in $j \in \hat{S}$, we threshold each element of $\hat{\pi}^{[j]}$ to create the thresholded estimate $\hat{\pi}^{[j]}$,

$$\hat{\pi}^{[j]}_k = \hat{\pi}^{[j]}_k 1 \left( k \in \hat{S} \cap |\hat{\pi}^{[j]}_k| \geq a_0 \sqrt{\hat{\Sigma}_{11}^{[j]} \frac{||\hat{V}_k - \hat{\gamma}^{[j]} \hat{V}_j||_2}{n}} \sqrt{\log \max(p_z, n)} \right)$$

for all $1 \leq k \leq p_z$. Each thresholded estimate $\hat{\pi}^{[j]}$ is obtained by looking at the elements of the un-thresholded estimate, $\hat{\pi}^{[j]}$, and examining whether each element exceeds the noise threshold (represented by the term $n^{-1} \sqrt{\Sigma_{11}^{[j]} ||\hat{V}_k - \hat{\gamma}^{[j]} \hat{V}_j||_2}$), adjusted for the multiplicity of the selection procedure (represented by the term $a_0 \sqrt{\log \max(p_z, n)}$). Among the $|\hat{S}|$ candidate estimates of $\pi$ based on each relevant instrument in $\hat{S}$, i.e. $\hat{\pi}^{[j]}$, we choose $\hat{\pi}^{[j]}$ with the most valid instruments, i.e. we choose $j^* \in \hat{S}$ where $j^* = \arg\min ||\hat{\pi}^{[j]}||_0$; if there is a non-unique solution, we choose $\hat{\pi}^{[j]}$ with the smallest $\ell_1$ norm, the closest convex norm of $\ell_0$. Subsequently, we can estimate the set of valid and relevant IVs, denoted as $\hat{V} \subseteq I$, as those elements of $\hat{\pi}^{[j^*]}$ that are zero,

$$\hat{V} = \hat{S} \setminus \text{supp}(\hat{\pi}^{[j^*]}) \ .$$

(31)
and estimate $\beta$ as

$$\hat{\beta}_E = \frac{\sum_{j \in \hat{V}} \hat{\gamma}_j \hat{\Gamma}_j}{\sum_{j \in \hat{V}} \hat{\gamma}_j^2}.$$  \hspace{1cm} (32)

The endogeneity test that is robust to invalid IVs has the same form as equation (22), except we use the set $\hat{V}$ instead of $\hat{S}$ and the estimate $\hat{\beta}_E$ instead of $\hat{\beta}$. We denote this endogeneity test as $Q_E$.

### 2.2 Properties of $Q_E$

We analyze the properties of the endogeneity test $Q_E$, which can handle invalid instruments as well as high dimensional instruments and covariates, even when $p > n$. Let $\mathcal{V} = \{ j \in \mathcal{I} \mid \pi_j = 0, \gamma_j \neq 0 \}$. We make the following assumptions that essentially control the behavior of selecting relevant and invalid IVs. We denote the assumption as “IN” since the assumption is specific to the case when invalid IVs are present.

1. **(IN1) (50% Rule)** The number of valid IVs is more than half of the number of non-redundant IVs, that is $|\mathcal{V}| > \frac{1}{2} |\mathcal{S}|$.

2. **(IN2) (Individual IV Strength)** Among IVs in $\mathcal{S}$, we have $\min_{j \in \mathcal{S}} |\gamma_j| \geq \delta_{\min} \gg \sqrt{\log p/n}$.

3. **(IN3) (Strong violation)** Among IVs in the set $\mathcal{S} \setminus \mathcal{V}$, we have

$$\min_{j \in \mathcal{S} \setminus \mathcal{V}} \left| \frac{\pi_j}{\gamma_j} \right| \geq \frac{12(1 + |\beta|)}{\delta_{\min}} \sqrt{\frac{M_1 \log \max\{p_x, n\}}{n \lambda_{\min}(\Theta)}}.$$  \hspace{1cm} (33)

In a nutshell, Assumption (IN1) states that if the number of invalid instruments is not too large, then we can use the observed data to separate the invalid IVs from valid IVs, without knowing a priori which IVs are valid or invalid. Assumption (IN1) is a relaxation of the assumption typical in IV settings where all the IVs are assumed to be valid a priori so that $|\mathcal{V}| = p_x$ and (IN1) holds automatically. In particular, Assumption (IN1) entertains the possibility that some IVs may be invalid, so $|\mathcal{V}| < p_x$, but without knowing a priori which IVs are invalid, i.e. the exact set $\mathcal{V}$. Assumption (IN1) is also the generalization of the 50% rule in Han (2008) and Kang et al. (2016) in the presence of redundant IVs. Also,
Kang et al. (2016) showed that this type of proportion-based assumption is a necessary component for identification of model parameters when instrument validity is uncertain.

Assumption (IN2) requires individual IV strength to be bounded away from zero. This assumption is needed to rule out IVs that are asymptotically weak. We also show in the simulation studies presented in the supplementary materials that (IN2) is largely unnecessary for our test to have proper size and have good power. Also, in the literature, (IN2) is similar to the “beta-min” condition assumption in high dimensional linear regression without IVs (Bühlmann and Van De Geer, 2011; Fan and Li, 2001; Wainwright, 2007; Zhao and Yu, 2006), with the exception that this condition is not imposed on our inferential quantity of interest, the endogeneity parameter $\Sigma_{12}$. Next, Assumption (IN3) requires the ratio $\pi_j/\gamma_j$ for invalid IVs to be large. This assumption is needed to correctly select valid IVs in the presence of possibly invalid IVs and this sentiment is echoed in the model selection literature by Leeb and Pötscher (2005) who pointed out that “in general no model selector can be uniformly consistent for the most parsimonious true model” and hence the post-model-selection inference is generally non-uniform (or uniform within a limited class of models). Specifically, for any IV with a small, but non-zero $|\pi_j/\gamma_j|$, such a weakly invalid IV is hard to distinguish from valid IVs where $\pi_j/\gamma_j = 0$. If a weakly invalid IV is mistakenly declared as valid, the bias from this mistake is of the order $\sqrt{\log p_k/n}$, which has consequences, not for consistency of the point estimation of $\Sigma_{12}$, but for a $\sqrt{n}$ inference of $\Sigma_{12}$; see the detailed discussion in Proposition 2 about point estimation of $\Sigma_{12}$ in Section 4.3 of the supplementary materials.

If all the instruments are valid, like the setting described in the majority of this paper where the IVs are valid conditional on many covariates, we do not need Assumptions (IN1)-(IN3) to make any claims about the proposed endogeneity test. However, in the presence of potentially invalid IVs that can grow in dimension, assumptions (IN1)-(IN3) are needed to control the behavior of the invalid IVs asymptotically and to characterize the asymptotic behavior of $Q_E$.

**Theorem 4.** Suppose we have models (2) and (3) where some instruments may be invalid, i.e. $\pi \neq 0$, and Assumptions (IN1)-(IN3) hold. If $\sqrt{C(V)} \gg s_{z1} \log p/\sqrt{n|V|}$, and
\( \sqrt{s_{z1}} \log p / \sqrt{n} \to 0, \) then for any \( \alpha, 0 < \alpha < 1, \) the Type I error of \( Q_E \) under \( H_0 \) is controlled at \( \alpha. \) Furthermore, the asymptotic power of \( Q \) under \( H_2 \) is equal to equation \( (24) \) with \( S = V. \)

Theorem 4 shows that our new test \( Q_E \) controls Type I error at the desired level \( \alpha. \) Also, Theorem 4 states that the power of \( Q_E \) is similar to the power of \( Q \) that knows exactly which instruments are valid and relevant. In short, our test \( Q_E \) is adaptive to the knowledge about instrument validity and can achieve similar level of performance as the test \( Q \) that knows about instrument validity a priori.

Finally, like Theorem 2, Theorem 4 controls the growth of the concentration parameter \( C(V) \) to be faster than \( s_{z1} \log p / \sqrt{n|V|}, \) with a minor discrepancy in the growth rate due to the differences between the sets \( V \) and \( S. \) But, as before, this growth condition is satisfied under the many instrument asymptotics of Bekker (1994) and the many weak instrument asymptotics of Chao and Swanson (2005). Also, like Theorem 4, the regularity conditions on \( s, s_{z1}, p, n \) are the same as those from Theorem 2.

3 Extended Simulation

3.1 Extended simulations for low dimensional instruments and covariates

In this section, we present all the simulation results from Section 6 in the main text for the low dimensional setting. These results are in Figure A1 to Figure A9. Across the figures, the column titled “WeakIV1” represents the case \( \rho_1 = 0.1, \) the column titled “WeakIV2” represents the case \( \rho_1 = 0.2 \) and the column titled with “Strong” represents the case \( \rho_1 = 0. \)

All the simulation results presented here match closely with the main simulation results presented in Section 6 in the main paper. In particular, when there are invalid instruments, (i) the regular DWH test cannot control Type I error and (ii) our proposed test performs similarly to the oracle DWH test. With regards to the latter, if \( \rho_2 = 0 \) so that all instruments are assumed to be valid and \( \rho_2 = 2 \) so that the instruments are strongly violated, the power curve of the proposed test is nearly identical to that of the oracle DWH test when \( n = 1000; \) see Figures A1, A3, A4, A6, A7 and A9. If \( \rho_2 = 1 \) so that the instruments are weakly invalid
and all the instruments are strong, the power curve of the proposed test is, again, nearly identical to that of the oracle DWH test when \( n = 1000 \); see the third column of Figure A2, A5 and A8. However, if \( \rho_2 = 1 \) and one of the instruments is relatively weak, a sample size of \( n = 10000 \) or even \( n = 50000 \) is required to empirically achieve asymptotic equivalence between our test and the oracle DWH test; see the first and second columns of Figure A2, A5 and A8.

### 3.2 Extended simulations for high dimensional instruments and covariates

In this section, we present all the simulation results from Section 6 in the main text for the high dimensional setting. The results are in Figures A10 to Figure A15. Across the figures, the column titled “WeakIV1” represents the case \( \rho_1 = 0.1 \), the column titled “WeakIV2” represents the case \( \rho_1 = 0.2 \) and the column titled with “Strong” represents the case \( \rho_1 = 0 \).

Again, all the simulation results presented here generally match with the main simulation results presented in Section 6 in the main paper. In particular, (i) the proposed test performs similarly to the oracle DWH test (see Figures A10 to A15), (ii) the regular DWH test suffers from relatively lower power in the presence of high dimensional covariates and/or instruments (see Figure A10), and (iii) the regular DWH test cannot control Type I error when invalid instruments are present (see Figures A11, A12, A14 and A15). Notice that across different columns in all the figures, our method has similar performance, which suggests that Assumption (IN2) is mainly a technical assumption and our method method is not very sensitive to the violation of this assumption. However, when assumption (IN3) is violated, which is represented by \( \rho_2 = 1 \), our test does not perform as well as the oracle DWH test when \( n = 300 \) and the concentration parameter hovers around \( C = 25 \); see the first row of Figure A11. However, with a larger concentration parameter, our test begins to perform similarly to the oracle DWH test and our Type I error is off by at most 5% to 10%. Also, when the sample size increases to 1000 and \( \rho_2 = 1 \), our proposed test is nearly identical to the oracle DWH test across different instrument strengths; see Figures A11 and A14.
Figure A1: Power of endogeneity tests when $n = 1000$, $\rho_2 = 0$, $p_x = 5$ and $p_z = 9$. The $x$-axis represents the endogeneity $\frac{\Sigma_{13}}{\sqrt{\Sigma_{11}\Sigma_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and $150$. 
Figure A2: Power of endogeneity tests when $n = 1000$, $\rho_2 = 1$, $p_x = 5$ and $p_z = 9$. The $x$-axis represents the endogeneity $\frac{\Sigma_{13}}{\sqrt{\Sigma_{11}\Sigma_{22}}}$ and the y-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and 150.
Figure A3: Power of endogeneity tests when $n = 1000$, $\rho_2 = 2$, $p_x = 5$ and $p_z = 9$. The $x$-axis represents the endogeneity $\frac{\Sigma_{13}}{\sqrt{\Sigma_{11} \Sigma_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and 150.
Figure A4: Power of endogeneity tests when $n = 10000$, $\rho_2 = 0$, $p_x = 5$ and $p_z = 9$. The $x$-axis represents the endogeneity $\frac{\Sigma_{13}}{\sqrt{\Sigma_{11}\Sigma_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and 150.
Figure A5: Power of endogeneity tests when $n = 10000$, $\rho_2 = 1$, $p_x = 5$ and $p_z = 9$. The $x$-axis represents the endogeneity $\frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and $150$. 
Figure A6: Power of endogeneity tests when $n = 10000$, $\rho_2 = 2$, $p_x = 5$ and $p_z = 9$. The $x$-axis represents the endogeneity $\frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and 150.
Figure A7: Power of endogeneity tests when \( n = 50000, \rho_2 = 0, p_x = 5 \) and \( p_z = 9 \). The \( x \)-axis represents the endogeneity \( \frac{\Sigma_{13}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \) and the \( y \)-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when \( \rho_1 = 0.1 \), \( \rho_1 = 0.2 \), and \( \rho_1 = 0 \). The rows represent different concentration parameters, \( nC(\mathcal{V}) = 25, 50, 75, 100 \) and 150.
Figure A8: Power of endogeneity tests when $n = 50000$, $\rho = 1$, $p_x = 5$ and $p_z = 9$. The $x$-axis represents the endogeneity $\Sigma_{12} / \sqrt{\Sigma_{11} \Sigma_{22}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and $150$. 
Figure A9: Power of endogeneity tests when \( n = 50000, \rho_2 = 2, p_x = 5 \) and \( p_z = 9 \). The \( x \)-axis represents the endogeneity \( \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \) and the \( y \)-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when \( \rho_1 = 0.1 \), \( \rho_1 = 0.2 \), and \( \rho_1 = 0 \). The rows represent different concentration parameters, \( nC(V) = 25, 50, 75, 100 \) and 150.
Figure A10: Power of endogeneity tests when $n = 300$, $\rho_2 = 0$, $p_x = 150$, and $p_z = 100$. The $x$-axis represents the endogeneity $\frac{\sum_{i=2}^2}{\sqrt{\sum_{i=1}^1 \sum_{i=2}^2}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test's empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and 150.
Figure A11: Power of endogeneity tests when $n = 300$, $\rho_2 = 1$, $p_x = 150$ and $p_z = 100$. The $x$-axis represents the endogeneity $\frac{\sum_{12}}{\sqrt{\sum_{11} \sum_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and 150.
Figure A12: Power of endogeneity tests when $n = 300$, $\rho_2 = 2$, $p_x = 150$ and $p_z = 100$. The $x$-axis represents the endogeneity $\frac{\Sigma_{12}}{\sqrt{\Sigma_{11} \Sigma_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test's empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(\mathcal{V}) = 25, 50, 75, 100$ and $150$. 
Figure A13: Power of endogeneity tests when $n = 1000$, $\rho_2 = 0$, $p_x = 150$ and $p_z = 100$. The $x$-axis represents the endogeneity $\sqrt{\Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and 150.
Figure A14: Power of endogeneity tests when $n = 1000$, $\rho_2 = 1$, $p_x = 150$ and $p_z = 100$. The $x$-axis represents the endogeneity $\sqrt{\frac{\Sigma_{12}}{\Sigma_{11} \Sigma_{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(V) = 25, 50, 75, 100$ and 150.
Figure A15: Power of endogeneity tests when $n = 1000$, $\rho_2 = 2$, $p_x = 150$ and $p_z = 100$. The $x$-axis represents the endogeneity $\frac{\sum_{i=2}^{12}}{\sqrt{\sum_{j=11}^{12} \sum_{k=22}^{22}}}$ and the $y$-axis represents the empirical power over 500 simulations. Each line represents a particular test’s empirical power over various values of the endogeneity. The columns “Weak IV1”, “Weak IV2”, and “Strong IV” represent the cases when $\rho_1 = 0.1$, $\rho_1 = 0.2$, and $\rho_1 = 0$. The rows represent different concentration parameters, $nC(\mathcal{V}) = 25, 50, 75, 100$ and 150.
4 Proof

4.1 Proof of Theorem 3

Assuming that \((\delta_i, \epsilon_i) \sim N\left(0, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)\), we have the following decomposition,

\[
\delta_i = \frac{\Sigma_{12}}{\Sigma_{22}} \epsilon_i + \tau_i,
\]

(34)

where \(\tau_i\) is independent of \(\epsilon_i\). By plugging (34) into (2) in the main paper, we have

\[
Y_i = D_i \beta + Z_i' \pi + X_i' \phi + \frac{\Sigma_{12}}{\Sigma_{22}} \epsilon_i + \tau_i.
\]

Let \(\sigma^2_\tau\) denote the variance of \(\tau_i\) and then \(\sigma_\tau = \sqrt{\Sigma_{11} - \frac{\Sigma_{12}^2}{\Sigma_{22}}}\). Define

\[
a_0(n) = \frac{\sigma_\tau}{\sqrt{\Sigma_{11}}} - 1 = \frac{-\Sigma_{12}^2}{\Sigma_{11} \Sigma_{22}} \sqrt{1 - \frac{\Sigma_{12}^2}{\Sigma_{11} \Sigma_{22}} + 1},
\]

(35)

then

\[
|a_0(n)| \leq C \frac{1}{n}.
\]

(36)

Note that

\[
\hat{\beta}_{OLS} = \beta + (D'P_{X\perp} D)^{-1} D'P_{X\perp} (Z, \epsilon) \left(\begin{pmatrix} \pi \\ \frac{\Sigma_{12}}{\Sigma_{22}} \end{pmatrix}\right) + (D'P_{X\perp} D)^{-1} D'P_{X\perp} \tau
\]

and

\[
\hat{\beta}_{TLS} = \beta + (D'(P_W - P_X) D)^{-1} D'(P_W - P_X) (Z, \epsilon) \left(\begin{pmatrix} \pi \\ \frac{\Sigma_{12}}{\Sigma_{22}} \end{pmatrix}\right)
\]

\[
+ (D'(P_W - P_X) D)^{-1} D'(P_W - P_X) \tau.
\]
We decompose the difference $\hat{\beta}_{TSL} - \hat{\beta}_{OLS}$ as

$$\hat{\beta}_{TSL} - \hat{\beta}_{OLS} = ((D'(P_W - P_X)D)^{-1} - (D'P_{X⊥}D)^{-1}) D'P_{X⊥}Z\pi$$

$$+ ((D'(P_W - P_X)D)^{-1}D'(P_W - P_X) - (D'P_{X⊥}D)^{-1}D'P_{X⊥})\epsilon^{\Sigma_{12}}_{\Sigma_{22}} (37)$$

$$+ ((D'(P_W - P_X)D)^{-1}D'(P_W - P_X) - (D'P_{X⊥}D)^{-1}D'P_{X⊥})\tau.$$  

Conditioning on $\epsilon$ and $W$, we have

$$L_1 = \frac{((D'(P_W - P_X)D)^{-1}D'(P_W - P_X) - (D'P_{X⊥}D)^{-1}D'P_{X⊥})\tau}{\sqrt{(D'(P_W - P_X)D)^{-1}\sigma_{22}^2}} \sim N(0, 1). (38)$$

By the assumption $\text{Cov}(W_i) = \Lambda$, $\text{Cov}\left(\delta_i, \epsilon_i\right) = \Sigma$ and weak law of large number, we have

$$\frac{1}{n}Z'Z \overset{p}{\to} \Lambda_{zz}, \quad \frac{1}{n}X'Z \overset{p}{\to} \Lambda_{xz}, \quad \frac{1}{n}X'X \overset{p}{\to} \Lambda_{xx},$$

$$\frac{1}{n}\epsilon'Z \overset{p}{\to} 0, \quad \frac{1}{n}\epsilon'X \overset{p}{\to} 0, \quad \frac{1}{n}\epsilon'\epsilon \overset{p}{\to} \Sigma_{22}.$$

Hence, we have

$$\left(\frac{1}{n}D'P_{X⊥}D\right)^{-1} \overset{p}{\to} (\gamma'\Lambda_{T|x;\gamma} + \Sigma_{22})^{-1}, \quad \left(\frac{1}{n}D'(P_W - P_X)D\right)^{-1} \overset{p}{\to} (\gamma'\Lambda_{T|x;\gamma})^{-1}, (39)$$

$$\frac{1}{n}D'P_{X⊥}Z\pi \overset{p}{\to} \gamma'\Lambda_{T|x;\pi}, \quad \frac{1}{n}D'(P_W - P_X)\epsilon \overset{p}{\to} 0 \quad \frac{1}{n}D'P_{X⊥}\epsilon \overset{p}{\to} \Sigma_{22}, (40)$$

By (39) and (40) and the parametrization $\Sigma_{12} = \frac{\Delta_1}{\sqrt{n}}$, we have

$$L_2 = \frac{\left(\frac{1}{n}D'(P_W - P_X)D - (D'P_{X⊥}D)^{-1}D'P_{X⊥}\right)\epsilon^{\Sigma_{12}}_{\Sigma_{22}}}{\sqrt{(D'(P_W - P_X)D)^{-1}\sigma_{22}^2}},$$

$$\overset{p}{\to} L_2^* = \Delta_1 \sqrt{\frac{\gamma'\Lambda_{T|x;\gamma}}{\Sigma_{11}\Sigma_{22}(\gamma'\Lambda_{T|x;\gamma} + \Sigma_{22})}} (41)$$

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By (39) and (40) and the parametrization \(\pi = \frac{\Delta_2}{\sqrt{n}}\) where \(\Delta_2\) is fixed vector, we have

\[
L_3 = \left(\frac{(D'(P_W - P_X)D)^{-1} - (D'P_XD)^{-1}}{\sqrt{(D'(P_W - P_X)D)^{-1}\sigma_r^2 - (D'P_XD)^{-1}\sigma_r^2}} \right) D'P_X \perp Z\pi
\]

With \(P \rightarrow L_3^* = \frac{\gamma' \Lambda_{\mathcal{I}|\mathcal{X}|\gamma} \Delta_2 \sqrt{\Sigma_{22}}}{\sqrt{\left(\gamma' \Lambda_{\mathcal{I}|\mathcal{X}|\gamma} + \Sigma_{22}\right) \left(\gamma' \Lambda_{\mathcal{I}|\mathcal{X}|\gamma}\right) \sqrt{\Sigma_{11}}}}\). 

Based on the above discussion, we can derive the general power curve as follows,

\[
P\left((L_1 + L_2 + L_3)^2 \geq \chi^2_\alpha(1)\right) = P\left(L_1 + L_2 + L_3 \geq \sqrt{\chi^2_\alpha(1)}\right) + P\left(L_1 + L_2 + L_3 \leq -\sqrt{\chi^2_\alpha(1)}\right)
\]

\[
= P\left(L_1 \geq \sqrt{\chi^2_\alpha(1)} - L_2 - L_3\right) + P\left(L_1 \leq -\sqrt{\chi^2_\alpha(1)} - L_2 - L_3\right)
\]

\[
= E_{W,\epsilon}\left(P\left(L_1 \geq \sqrt{\chi^2_\alpha(1)} - L_2 - L_3 | W, \epsilon\right) + P\left(L_1 \leq -\sqrt{\chi^2_\alpha(1)} - L_2 - L_3 | W, \epsilon\right)\right).
\]

By (38), conditioning on \(W\) and \(\epsilon\), we have

\[
P\left(L_1 \geq \sqrt{\chi^2_\alpha(1)} - L_2 | W, \epsilon\right) = 1 - \Psi\left(\frac{\sqrt{\chi^2_\alpha(1)} - L_2 - L_3}{1 + a_0(n)}\right),
\]

\[
P\left(L_1 \leq -\sqrt{\chi^2_\alpha(1)} - L_2 | W, \epsilon\right) = \Psi\left(\frac{-\sqrt{\chi^2_\alpha(1)} - L_2 - L_3}{1 + a_0(n)}\right).
\]

Combined with (36), (38), (45) and (46), we establish (29) in the main paper. The type I error control (27) follows from (28) with taking \(\Delta_2 = 0\).

For the case \(0 \leq k < \frac{1}{2}\), we apply the similar argument as that of (28) to establish (26) and the only difference is that

\[
L_3 \xrightarrow{P} \frac{\sqrt{\Sigma_{22}}}{\sqrt{\left(\gamma' \Lambda_{\mathcal{I}|\mathcal{X}|\gamma} + \Sigma_{22}\right) \left(\gamma' \Lambda_{\mathcal{I}|\mathcal{X}|\gamma}\right) \sqrt{\Sigma_{11}}}}.
\]

As \(\gamma' \Lambda_{\mathcal{I}|\mathcal{X}|\gamma} \Delta_2 \rightarrow \infty\), we establish (26) in the main paper. For the case \(k > \frac{1}{2}\), we apply the similar argument as that of (28) and we can establish (29) with the fact \(L_3 \xrightarrow{P} 0\).
4.2 Proof of Theorem 1

By (37), we have the following expression of \( \hat{\beta}_{TLS} - \hat{\beta}_{OLS} \),

\[
\hat{\beta}_{TLS} - \hat{\beta}_{OLS} = ((D'(P_W - P_X)D)^{-1}D'(P_W - P_X) - (D'P_{X\perp}D)^{-1}D'P_{X\perp}) \epsilon \frac{\Sigma_{12}}{\Sigma_{22}} \\
+ ((D'(P_W - P_X)D)^{-1}D'(P_W - P_X) - (D'P_{X\perp}D)^{-1}D'P_{X\perp}) \tau.
\]

Hence, the test statistic \( Q_{DWH} \) has the following expression,

\[
Q_{DWH} = \frac{(\hat{\beta}_{TLS} - \hat{\beta}_{OLS})^2}{(D'(P_W - P_X)D)^{-1}1_{11} - (D'P_{X\perp}D)^{-1}1_{11}} = (L_1 + L_2)^2,
\]

where

\[
L_1 = \frac{((D'(P_W - P_X)D)^{-1}D'(P_W - P_X) - (D'P_{X\perp}D)^{-1}D'P_{X\perp}) \epsilon \frac{\Sigma_{12}}{\Sigma_{22}}}{\sqrt{(D'(P_W - P_X)D)^{-1}1_{11} - (D'P_{X\perp}D)^{-1}1_{11}}} \times \frac{\sigma_\tau}{\sqrt{\Sigma_{11}}}
\]

and

\[
L_2 = \frac{((D'(P_W - P_X)D)^{-1}D'(P_W - P_X) - (D'P_{X\perp}D)^{-1}D'P_{X\perp}) \epsilon \frac{\Sigma_{12}}{\Sigma_{22}}}{\sqrt{(D'(P_W - P_X)D)^{-1}1_{11} - (D'P_{X\perp}D)^{-1}1_{11}}} \times \frac{\sigma_\tau}{\sqrt{\Sigma_{11}}}
\]

Since \( W_i \) is a zero-mean multivariate Gaussian, we have \( Z'_i = X'_i \left( \Lambda_{xx}^{-1} \Lambda_{xz} \right) + Z'_i \), where \( Z'_i \) is independent of \( X_i \) and \( Z_i \) is of mean 0 and covariance matrix \( \Lambda_{XX} \). Hence, we have \( D = Z \gamma + X (\psi + \Lambda_{xx}^{-1} \Lambda_{xz} \gamma) + \epsilon \). Note that \( Z_i \gamma \sim N(0, \gamma' \Lambda_{XX} \epsilon) \), then we have the following decomposition,

\[
\frac{1}{n} D'(P_W - P_X) \epsilon = \frac{1}{n} (Z \gamma + X (\psi + \Lambda_{xx}^{-1} \Lambda_{xz} \gamma) + \epsilon)' \left( W(W'W)^{-1} W' - X(X'X)^{-1} X' \right) \epsilon \\
= \frac{1}{n} \gamma' Z_i' \left( 1 - X(X'X)^{-1} X' \right) \epsilon + \frac{1}{n} \epsilon' \left( W(W'W)^{-1} W' - X(X'X)^{-1} X' \right) \epsilon.
\]
Define

$$L_{2,1} = \frac{\sqrt{n} \Sigma_{12}}{\Sigma_{22} \sqrt{\Sigma_{11}}} \sqrt{\left( \frac{1}{n} D'(P_W - P_X)D \right)^{-1} - \left( \frac{1}{n} D'P_{X^\perp} D \right)^{-1}}$$

$$L_{2,2} = \frac{1}{n} \gamma' \tilde{Z}' \left( 1 - X (X'X)^{-1} X' \right) \epsilon$$

$$L_{2,3} = \frac{1}{n} \epsilon' \left( W (W'W)^{-1} W' - X (X'X)^{-1} X' \right) \epsilon$$

$$L_{2,4} = \frac{\sqrt{n} \Sigma_{12}}{\Sigma_{22} \sqrt{\Sigma_{11}}} \left( \frac{1}{n} D' P_{X^\perp} P_{X^\perp} - \left( \frac{1}{n} D' P_{W - P_X} \right) \epsilon \right) \sqrt{\left( \frac{1}{n} D' (P_W - P_X)D \right)^{-1} - \left( \frac{1}{n} D' P_{X^\perp} D \right)^{-1}}$$

and we have

$$L_2 = L_{2,1} \times (L_{2,2} + L_{2,3}) - L_{2,4}.$$ (47)

For the term $L_{2,2}$, conditioning on $X$, we have

$$L_{2,2} = \frac{1}{n} \sum_{i=1}^{n-p_x} U_i V_i = \frac{p_k}{n} \frac{1}{n} \frac{1}{p_x} \sqrt{n - p_x} \sqrt{\frac{\gamma' \Lambda_{I|I^c} \gamma}{p_x \Sigma_{22}}} \left( \frac{1}{\sqrt{n - p_x}} \sum_{i=1}^{n-p_x} \frac{U_i}{\sqrt{\gamma' \Lambda_{I|I^c} \gamma} \Sigma_{22}} V_i \right),$$

where $U_i$ is independent of $V_i$, $U_i$ follows i.i.d normal with mean 0 and variance $\gamma' \Lambda_{I|I^c} \gamma$ and $V_i$ follows i.i.d normal with mean 0 and variance $\Sigma_{22}$.

Define

$$L^*_{2,1} = \frac{\sqrt{n} \Sigma_{12}}{\Sigma_{22} \sqrt{\Sigma_{11}}} \sqrt{\left( \frac{n-p_x}{n} \left( \gamma' \Lambda_{I|I^c} \gamma + \Sigma_{22} \right) \right) \left( \frac{n-p_x}{n} \left( \gamma' \Lambda_{I|I^c} \gamma + \Sigma_{22} \right) \right)}$$

$$= \frac{\sqrt{n} \Sigma_{12}}{\Sigma_{22} \sqrt{\Sigma_{11}}} \sqrt{\left( \frac{n-p_x}{n} \left( \gamma' \Lambda_{I|I^c} \gamma + \Sigma_{22} \right) \right) \left( \frac{n-p_x}{n} \left( \gamma' \Lambda_{I|I^c} \gamma + \Sigma_{22} \right) \right)}$$

$$L^*_{2,3} = \frac{p_k}{n} \Sigma_{22}$$

$$L^*_{2,4} = \frac{\sqrt{n} \Sigma_{12}}{\Sigma_{22} \sqrt{\Sigma_{11}}} \left( 1 - \frac{p_x}{n} \right) \Sigma_{22} \left( \frac{\gamma' \Lambda_{I|I^c} \gamma + \Sigma_{22}}{p_x \Sigma_{22}} + \frac{1}{n-p_x} \right)$$

The following Lemma characterizes the difference between $L_{2,i}$ and $L^*_{2,i}$ for $i = 1, 3, 4$.

**Lemma 1.** Define

$$a_1(n) = \frac{L_{2,1}}{L^*_{2,1}} - 1, \quad \text{and} \quad a_2(n) = \frac{L_{2,3}}{L^*_{2,3}} - 1, \quad \text{and} \quad a_3(n) = \frac{L_{2,4}}{L^*_{2,4}} - 1.$$ (48)
Then there exists an event $A$ such that

$$P(A) \geq 1 - (\min\{n - p, p_x\})^{-c}. \quad (49)$$

and on the event $A$, we have

$$\max\{|a_1(n)|, |a_3(n)|\} \leq C \left( \frac{\log p_z}{p_z} + \frac{\log(n - p)}{n - p} + \sqrt{\frac{\log(n - p_x)}{n - p_x}} \sqrt{\frac{\Sigma_{22}}{\gamma' \Lambda_I \{I \cap \gamma\}}} \right), \quad (50)$$

and

$$|a_2(n)| \leq C \frac{\log p_z}{p_z}, \quad (51)$$

for some positive constant $C$.

By (44), we derive the power function as follows,

$$P \left( (L_1 + L_2)^2 \geq \chi^2_{\alpha}(1) \right) = P \left( L_1 + L_2 \geq \sqrt{\chi^2_{\alpha}(1)} \right) + P \left( L_1 + L_2 \leq -\sqrt{\chi^2_{\alpha}(1)} \right)$$

$$= P \left( L_1 \geq \sqrt{\chi^2_{\alpha}(1)} - L_2 \right) + P \left( L_1 \leq -\sqrt{\chi^2_{\alpha}(1)} - L_2 \right) \quad (52)$$

$$= E_{W, \epsilon} \left( P \left( L_1 \geq \sqrt{\chi^2_{\alpha}(1)} - L_2 \mid W, \epsilon \right) + P \left( L_1 \leq -\sqrt{\chi^2_{\alpha}(1)} - L_2 \mid W, \epsilon \right) \right).$$

By (46) and (45), conditioning on $W$ and $\epsilon$, we have

$$P \left( L_1 \geq \sqrt{\chi^2_{\alpha}(1)} - L_2 \mid W, \epsilon \right) = 1 - \Psi \left( \frac{\sqrt{\chi^2_{\alpha}(1)} - L_2}{1 + a_0(n)} \right),$$

$$P \left( L_1 \leq -\sqrt{\chi^2_{\alpha}(1)} - L_2 \mid W, \epsilon \right) = \Psi \left( \frac{-\sqrt{\chi^2_{\alpha}(1)} - L_2}{1 + a_0(n)} \right). \quad (53)$$

By (47), (52) and (53), we have

$$P \left( (L_1 + L_2)^2 \geq \chi^2_{\alpha}(1) \right) = E_{W, \epsilon} \left( 1 - \Psi \left( \frac{-\sqrt{\chi^2_{\alpha}(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4}}{1 + a_0(n)} \right) \right)$$

$$+ E_{W, \epsilon} \Psi \left( \frac{\sqrt{\chi^2_{\alpha}(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4}}{1 + a_0(n)} \right). \quad (54)$$

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We have the following decomposition of the difference,

\[
\left| E_{W, \epsilon} \left( \sqrt{\chi_0^2(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4} \right) - \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}^*L_{2,3} + L_{2,4}^* \right) \right| \\
\leq E_{W, \epsilon} \left( \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4} \right) - \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}^*L_{2,3} + L_{2,4}^* \right) \right) \cdot 1_A \\
+ E_{W, \epsilon} \left( \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4} \right) - \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}^*L_{2,3} + L_{2,4}^* \right) \right) \cdot 1_{A^c}
\]

where \(A\) is defined in Lemma 1. Since \(\sup_{x \in \mathbb{R}} |\Psi(x)| \leq 1\), we have

\[
\left| E_{W, \epsilon} \left( \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4} \right) - \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}^*L_{2,3} + L_{2,4}^* \right) \right) \right| \leq P(A^c). \tag{55}
\]

It is remaining to control

\[
\left| E_{W, \epsilon} \left( \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4} \right) - \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}^*L_{2,3} + L_{2,4}^* \right) \right) \right| \cdot 1_A.
\]

By the fact that \(\sup_{x \in \mathbb{R}} |\Psi'(x)| \leq 1\), we have

\[
\left| E_{W, \epsilon} \left( \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4} \right) - \Psi \left( \sqrt{\chi_0^2(1)} - L_{2,1}^*L_{2,3} + L_{2,4}^* \right) \right) \right| \cdot 1_A \\
\leq E_{W, \epsilon} \left| \sqrt{\chi_0^2(1)} - L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4} - \left( \sqrt{\chi_0^2(1)} - L_{2,1}^*L_{2,3} + L_{2,4}^* \right) \right| \cdot 1_A \\
\leq \left| \frac{a_0(n)}{1 + a_0(n)} \right| \sqrt{\chi_0^2(1)} + \left| L_{2,1}^* \right| \times E_{W, \epsilon} \left( \left| 1 - \frac{L_{2,1}}{L_{2,1}^*} \right| + 1 \right) |L_{2,2}| \cdot 1_A \\
+ \left| L_{2,1}^*L_{2,3} \right| \times E_{W, \epsilon} \left| 1 - \frac{L_{2,1}L_{2,3}}{L_{2,1}^*L_{2,3}^* + 1 + a_0(n)} \right| \cdot 1_A + \left| L_{2,4}^* \right| \times E_{W, \epsilon} \left| 1 - \frac{L_{2,4}}{L_{2,4}^*} \right| \cdot 1_A \tag{56}
\]

where the last inequality follows from triangle inequality. By the definition (48), the last term in the above inequality can be expressed as

\[
\left| \frac{a_0(n)}{1 + a_0(n)} \right| \sqrt{\chi_0^2(1)} + \left| L_{2,1}^* \right| \times E_{W, \epsilon} \left( \left| 1 - \frac{1 + a_1(n)}{1 + a_0(n)} \right| + 1 \right) \cdot |L_{2,2}| \cdot 1_A \\
+ \left| L_{2,1}^*L_{2,3} \right| \times E_{W, \epsilon} \left| 1 - \frac{(1 + a_1(n))(1 + a_2(n))}{1 + a_0(n)} \right| \cdot 1_A + \left| L_{2,4}^* \right| \times E_{W, \epsilon} \left| 1 - \frac{1 + a_3(n)}{1 + a_0(n)} \right| \cdot 1_A, \tag{57}
\]

which the last inequality follows the fact \(|a_0(n)| \leq C \frac{1}{n}\) and Lemma 1. By the parametrization
\[ \Sigma_{12} = \frac{\Delta_1}{\sqrt{n}}, \] we have \( |L_{2,4}^*| \leq \frac{\Delta_1}{\sqrt{\Sigma_{11}\Sigma_{22}}} \) and

\[
|L_{2,1}^*| \cdot |L_{2,3}^*| = \frac{\Delta_1}{\sqrt{\Sigma_{11}\Sigma_{22}}} \sqrt{\frac{n-p}{(n-p_x)p_x} \left( \frac{\gamma'\Lambda_{12|x}\gamma}{p_x\Sigma_{22}} + \frac{1}{n-p_x} \right) \left( \frac{\gamma'\Lambda_{12|x}\gamma}{p_x\Sigma_{22}} + \frac{1}{p_x} \right)}
\]

and

\[
|L_{2,1}^*| E_{W,\epsilon} |L_{2,2}| = |L_{2,1}^*| \cdot |L_{2,3}^*| E \left\{ \sqrt{\frac{n-p_x}{p_x}} \sqrt{\frac{\gamma'\Lambda_{12|x}\gamma}{p_x\Sigma_{22}}} |S_n| \right\}
\]

\[
\leq C \frac{\Delta_1}{\sqrt{\Sigma_{11}\Sigma_{22}}} \sqrt{\frac{n-p}{(n-p_x)p_x} \left( \frac{\gamma'\Lambda_{12|x}\gamma}{p_x\Sigma_{22}} + \frac{1}{n-p_x} \right) \left( \frac{\gamma'\Lambda_{12|x}\gamma}{p_x\Sigma_{22}} + \frac{1}{p_x} \right)}
\]

By the fact that \( (n-p_x)n \left( \frac{\gamma'\Lambda_{12|x}\gamma}{p_x\Sigma_{22}} + \frac{1}{n-p_x} \right) \left( \frac{\gamma'\Lambda_{12|x}\gamma}{p_x\Sigma_{22}} + \frac{1}{p_x} \right) \rightarrow \infty \) and \( p_xn \left( \frac{\gamma'\Lambda_{12|x}\gamma}{p_x\Sigma_{22}} + \frac{1}{p_x} \right) \rightarrow \infty \), we have

\[
|L_{2,1}^*| \cdot |L_{2,3}^*| \rightarrow 0 \quad \text{and} \quad |L_{2,1}^*| E_{W,\epsilon} |L_{2,2}| \rightarrow 0.
\] (58)

Since \( \sqrt{C(T)} \gg \sqrt{\log(n-p_x)/(n-p_x)p_x} \), combined with (56), (57) and (58), we establish that

\[
\left| E_{W,\epsilon} \left( \Psi \left( \sqrt{\frac{X_0^2(n)}{1+a_0(n)}} \frac{L_{2,1}(L_{2,2} + L_{2,3}) + L_{2,4}}{1 + a_0(n)} \right) - \Psi \left( \sqrt{\frac{X_0^2(n)}{1+a_0(n)}} L_{2,1}^* L_{2,3}^* + L_{2,4}^* \right) \right) \right| \cdot 1_A \rightarrow 0.
\] (59)

By (55),(59) and Lemma 1, we establish (10) in the main paper.

### 4.3 Proof of Theorem 4

We introduce the notation \( \Pi = (\xi, \epsilon) \in \mathbb{R}^{n \times 2} \). We introduce the following proposition, which is Theorem 2 in Guo et al. (2016a).

**Proposition 1.** Under the same assumptions as Theorem 4, the following property holds for the estimator \( \hat{\beta} \) defined in (32) in the main paper,

\[
\sqrt{n} \left( \hat{\beta} - \beta \right) = T^\beta + \Delta^\beta
\] (60)

with

\[
T^\beta = \frac{1}{\sum_{j \in V} \gamma_j} \sum_{j \in V} \gamma_j (\overline{V}_j)' (\Pi_1 - \beta \Pi_2),
\]
and
\[ \limsup_{n \to \infty} P \left( \left| \frac{\Delta^\beta}{\sqrt{\Var_1}} \right| \geq C \left( \sqrt{s_{z1} \log p} \frac{1}{\sqrt{n}} + \frac{s_{z1} \log p}{\| \gamma \|_2 \sqrt{n}} \right) \right) = 0, \]

where \( \Var_1 = \frac{1}{\sum_{j \in V} \gamma_j^2} \times \left\| \sum_{j \in V} \gamma_j V_j \right\|_2^2 \left( \Theta_{11} + \beta^2 \Theta_{22} - 2 \beta \Theta_{12} \right). \) Note that \( T^\beta | W \sim N(0, \Var_1) \).

Recall the definition \( \hat{\Sigma}_{12} = \hat{\Theta}_{12} - \hat{\beta} \hat{\Theta}_{22} \). We have the following expression for \( \hat{\Sigma}_{12} - \Sigma_{12} \),
\[
\hat{\Sigma}_{12} - \Sigma_{12} = (\hat{\Theta}_{12} - \Theta_{12}) - (\hat{\Theta}_{12} - \beta \Theta_{22}) - (\hat{\beta} - \beta) \Theta_{22} - (\hat{\beta} - \beta) (\hat{\Theta}_{22} - \Theta_{22}).
\]

Define \( v = \sum_{j \in V} \gamma_j V_j \) and let \( P_v \) denote the projection matrix to the direction of \( v \) and \( P_v^\perp \) denote the projection matrix to the orthogonal complement of \( v \), that is, \( P_v = v (v'v)^{-1} v' \) and \( P_v^\perp = I - v (v'v)^{-1} v' \). Define
\[
\Delta^\Theta_{11} = \sqrt{n} \left( \hat{\Theta}_{11} - \frac{1}{n} \Pi_1^' \Pi_1 \right), \quad \Delta^\Theta_{12} = \sqrt{n} \left( \hat{\Theta}_{12} - \frac{1}{n} \Pi_1^' \Pi_2 \right),
\]
\[
\Delta^\Theta_{22} = \sqrt{n} \left( \hat{\Theta}_{22} - \frac{1}{n} \Pi_2^' \Pi_2 \right).
\]

and
\[
\sqrt{n} \left( \hat{\Theta}_{12} - \Theta_{12} \right) = \frac{1}{\sqrt{n}} \left( \langle \Pi_1^' P_v^\perp \Pi_2 - (n-1) \Theta_{12} \rangle \right) + \Delta^\Theta_{12},
\]
\[
\sqrt{n} \left( \hat{\Theta}_{22} - \Theta_{22} \right) = \frac{1}{\sqrt{n}} \left( \langle \Pi_2 P_v^\perp \Pi_2 - (n-1) \Theta_{22} \rangle \right) + \Delta^\Theta_{22},
\]

Hence, we can further decompose \( \sqrt{n} \left( \hat{\Sigma}_{12} - \Sigma_{12} \right) \) as
\[
\sqrt{n} \left( \hat{\Sigma}_{12} - \Sigma_{12} \right) = M_1 + M_2 + R_1 + R_2 + R_3,
\]

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where

\[ M_1 = -\Theta_{22} \frac{1}{\sum_{j \in \mathcal{V}} \gamma_j} \sum_{j \in \mathcal{V}} \gamma_j (\hat{V}_j)' (\Pi_1 - \beta \Pi_2), \]

\[ M_2 = \frac{1}{\sqrt{n}} \left( (\Pi'_1 P_{v+} \Pi_2 - (n - 1) \Theta_{12}) - \beta (\Pi'_2 P_{v+} \Pi_2 - (n - 1) \Theta_{22}) \right), \]

\[ R_1 = -\Theta_{22} \Delta \beta + \Delta \Theta_{12} - \beta \Delta \Theta_{22}, \]

\[ R_2 = \frac{1}{\sqrt{n}} \left( (\Pi'_1 P_{v} \Pi_2 - \Theta_{12}) - \beta (\Pi'_2 P_{v} \Pi_2 - \Theta_{22}) \right), \]

\[ R_3 = \sqrt{n} \left( \bar{\beta} - \beta \right) \left( \hat{\Theta}_{22} - \Theta_{22} \right). \]

By rescaling, we have

\[
\sqrt{n} \frac{\hat{\Sigma}_{12} - \Sigma_{12}}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} = \frac{M_1 + M_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + \frac{R_1 + R_2 + R_3}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}}. \tag{62}
\]

Define the following events,

\[ B_1 = \left\{ \left| \frac{\Delta \beta}{\sqrt{\text{Var}_1}} \right| \leq C \left( \frac{s \log p}{\sqrt{n}} + \frac{1}{s \log p} \frac{s_{z1} \log p}{\sqrt{n}} \right) \right\}, \]

\[ B_2 = \left\{ \left| \bar{\beta} - \beta \right| \leq C \sqrt{\frac{\log p}{n} \sqrt{V_1}} \right\}, \]

\[ B_3 = \left\{ \max |\tilde{\gamma}_j - \gamma_j| \leq C \sqrt{\frac{\log p}{n}} \right\}, \]
and
\[ B_4 = \left\{ \max\{|\Delta^{\Theta_{12}}|, |\Delta^{\Theta_{22}}|, |\Delta^{\Theta_{11}}|\} \leq C \frac{s \log p}{\sqrt{n}} \right\}, \]
\[ B_5 = \left\{ \max_{1 \leq i, j \leq 2} |\hat{\Theta}_{ij} - \Theta_{ij}| \leq \sqrt{\frac{\log p}{n}} \right\}, \]
\[ B_6 = \left\{ c_0 \leq \min_{1 \leq j \leq p_s} \|\hat{V}_j\|_2 \leq \max_{1 \leq j \leq p_s} \|\hat{V}_j\|_2 \leq C_0 \right\}, \]
\[ B_7 = \left\{ \|\sum_{j \in V} \gamma_j \hat{V}_j\|_2 \geq c_1 \|\gamma\|_2 \right\}, \]
\[ B_8 = \left\{ \frac{c_1^2}{\|\gamma\|_2^2} \leq \frac{1}{\|\gamma\|_2^2} \left\|\sum_{j \in V} \gamma_j \hat{V}_j\right\|_2^2 \leq \frac{C_0^2 s_{z_1}}{\|\gamma\|_2^2} \right\}, \]
\[ B_9 = \left\{ \frac{1}{\sqrt{n}} \left|\left(\Pi_1^T P_2 \Pi_2 - \Theta_{12}\right) - \beta \left(\Pi_2^T P_1 \Pi_2 - \Theta_{22}\right)\right| \leq C \frac{\log p}{\sqrt{n}} \right\}. \]

Define \[ B = \cap_{i=1}^9 B_i. \] Note that \[ B_6 \cap B_7 \subset B_8. \] By Proposition 1 and the fact that \( (\hat{\gamma}, \hat{\Gamma}, \hat{\Theta}_{11}, \hat{\Theta}_{22}, \hat{\Theta}_{12}) \) is well-behaved estimator, we have
\[ \liminf_{n \to \infty} P(B) = 1. \] (64)

We introduce the following lemma.

**Lemma 2.** Under the same assumptions as Theorem 4, we can establish the following properties,
\[ M_1 \perp M_2 \mid W; \] (65)

On the event \( B \), we have
\[ \left| \frac{R_1 + R_2 + R_3}{\sqrt{\text{Var} \, 2 + \Theta_{22}^2 \text{Var} \, 1}} \right| \leq C \left( \frac{s \log p}{\sqrt{n}} + \frac{1}{\|\gamma\|_2} \frac{s_{z_1} \log p}{\sqrt{n}} \right), \] (66)

and
\[ \left| \sqrt{\frac{\text{Var}(\hat{\Sigma}_{12})}{\Theta_{22}^2 \text{Var} \, 1 + \text{Var} \, 2}} - 1 \right| \leq C \frac{1}{\sqrt{s_{z_1} \log p}}. \] (67)
We have the following expression,

\[
P \left( |Q_N| \geq z_{\alpha/2} \right) = P \left( \sqrt{n} \frac{\hat{\Sigma}_{12} - \Sigma_{12}}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + \sqrt{n} \frac{\hat{\Sigma}_{12}}{\sqrt{\text{Var}_2 + \text{Var}_1 + \text{Var}_2}} \geq z_{\alpha/2} \sqrt{\frac{\text{Var}(\hat{\Sigma}_{12})}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2}} \right)
\]

\[
= P \left( \frac{M_1 + M_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + \sqrt{n} \frac{\hat{\Sigma}_{12}}{\sqrt{\text{Var}_2 + \text{Var}_1 + \text{Var}_2}} \geq z_{\alpha/2} \frac{\sqrt{\text{Var}(\hat{\Sigma}_{12})}}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} - \frac{R_1 + R_2 + R_3}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \right)
\]

\[
+ P \left( \frac{M_1 + M_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + \sqrt{n} \frac{\hat{\Sigma}_{12}}{\sqrt{\text{Var}_2 + \text{Var}_1 + \text{Var}_2}} \leq -z_{\alpha/2} \frac{\sqrt{\text{Var}(\hat{\Sigma}_{12})}}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} - \frac{R_1 + R_2 + R_3}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \right).
\]

Define \( B(\theta, W) = \frac{\sqrt{n} \hat{\Sigma}_{12}}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \). We introduce the following lemma to approximate the power curve defined in (68).

**Lemma 3.** Under the same assumptions as Theorem 4, we have

\[
P \left( \frac{M_1 + M_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + B (\theta, W) \geq z_{\alpha/2} \frac{\sqrt{\text{Var}(\hat{\Sigma}_{12})}}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} - \frac{R_1 + R_2 + R_3}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \right)
\]

\[
- \left( 1 - E_W \Phi \left( z_{\alpha/2} - B (\theta, W) \right) \right) \to 0.
\]

and

\[
P \left( \frac{M_1 + M_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + B (\theta, W) \leq -z_{\alpha/2} \frac{\sqrt{\text{Var}(\hat{\Sigma}_{12})}}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} - \frac{R_1 + R_2 + R_3}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \right)
\]

\[
- \Phi \left( -z_{\alpha/2} - B (\theta, W) \right) \to 0.
\]

By (68) and Lemma 3, we establish (24) in the main paper. And (23) in the main paper follows from (24) in the main paper by taking \( \Sigma_{12} = 0 \).

In addition to testing \( \Sigma_{12} = 0 \), we can also establish the consistency of estimating \( \Sigma_{12} \).

**Proposition 2.** Suppose we have the models (2) and (3) where some instruments may be invalid, i.e. \( \pi \neq 0 \), and Assumptions (IN1)-(IN2) hold. If \( \sqrt{C(V)} \gg s_{z1} \log p/\sqrt{n|V|} \), and \( \sqrt{s_{z1}s} \log p/n \to 0 \), then

\[
\left| \hat{\Sigma}_{12} - \Sigma_{12} \right| = O_p \left( \frac{1}{\delta_{\min}} \sqrt{\frac{\log p_z}{n}} \right).
\]

The proof of the above Proposition is based on the expression (62), the condition (15) and the following Lemma, which is Theorem 2 in Guo et al. (2016a).
Lemma 4. Under the same assumptions as Proposition 2, with probability larger than 
\[ 1 - c \left( p^{-c} + s_{z1}^2 p^{-1.01} + \exp(-cn) \right), \]
\[ \left| \hat{\beta} - \beta^* \right| \leq C \frac{1}{\delta_{\min}} \sqrt{\log p_{z} \frac{\log p_{z}}{n}}, \]
where \( c, C > 0 \) are constants independent of \( n \) and \( p \).

4.4 Proof of Theorem 2

We first introduce the following proposition, which is Theorem 3 in Guo et al. (2016a).

Proposition 3. Under the same assumptions as Theorem 2, the following property holds
for the estimator \( \hat{\beta} \) defined in (21) in the main paper,
\[ \sqrt{n} \left( \hat{\beta} - \beta \right) = T^\beta + \Delta^\beta, \] (71)
with \( T^\beta = \frac{1}{\sum_{j \in S} \gamma_j^2} \sum_{j \in S} \gamma_j (\hat{V}_j)' (\Pi_1 - \beta \Pi_2) \) and
\[ \limsup_{n \to \infty} P \left( \left| \frac{\Delta^\beta}{\sqrt{\text{Var}_1}} \right| \geq C \left( \frac{s_{z1} s \log p}{\sqrt{n}} + \frac{1}{\|\gamma\|_2} \frac{s_{z1} \log p}{\sqrt{n}} \right) \right) = 0, \] (72)
where \( \text{Var}_1 = 1/\left( \sum_{j \in S} \gamma_j^2 \right)^2 \times \| \sum_{j \in S} \gamma_j \hat{V}_j \|_2^2 \left( \Theta_{11} + \beta^2 \Theta_{22} - 2\beta \Theta_{12} \right). \) Note that \( T^\beta \mid W \sim N(0, \text{Var}_1). \)

Applying the similar argument with the proof of Theorem 4 in Section 4.3, we can establish Theorem 2.

5 Proof of key lemmas

5.1 Proof of Lemma 1

We first introduce the following technical lemmas, which will be used to prove Lemma 1. The first lemma (Theorem 2.3 in Boucheron et al. (2013)) is a concentration result of \( \chi^2 \) random variable.
Lemma 5. Let $\chi^2_n$ denote the $\chi^2$ random variable with $n$ degrees of freedom, then we have the following concentration inequality,

$$
\Pr \left( |\chi^2_n - E\chi^2_n| > 2\sqrt{nt} + 2t \right) \leq 2 \exp(-t).
$$

The following lemma establishes the concentration of sum of independent centered subexponential random variables (Vershynin (2012) and Javanmard and Montanari (2014)),

Lemma 6. Let $X_i$ denote sub-exponential random variable with the sub-exponential norm $K = \|X_i\|_{\psi_1}$, then we have

$$
P \left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq \epsilon \right) \leq 2 \exp \left( -\frac{1}{6} n \min \left( \frac{\epsilon}{K}, \frac{\epsilon^2}{K^2} \right) \right).
$$

(73)

Note that $Z_i = X_i (\Lambda_{xx}^{-1} \Lambda_{xz}) + \bar{Z}_i$, where

$$
\begin{pmatrix}
\bar{Z}_i \\
\bar{X}_i
\end{pmatrix}
\sim
N
\left(
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\begin{pmatrix}
\Lambda_{xx} & 0 \\
0 & \Lambda_{xx}
\end{pmatrix}
\right)
$$

and hence

$$
D = \bar{Z}_\gamma + X (\psi + \Lambda_{xx}^{-1} \Lambda_{xz}) + \epsilon.
$$

Note that $\bar{Z}_i, \gamma \sim N(0, \gamma' \Lambda_{xx} \gamma)$. We have the following decompositions

$$
\frac{1}{n} D' P_X D
$$

$$
= \frac{1}{n} \left( \bar{Z}_\gamma + X (\psi + \Lambda_{xx}^{-1} \Lambda_{xz}) + \epsilon \right)' \left( I - X (X'X)^{-1} X' \right) \left( \bar{Z}_\gamma + X (\psi + \Lambda_{xx}^{-1} \Lambda_{xz}) + \epsilon \right)
$$

$$
= \frac{1}{n} \left( \bar{Z}_\gamma + \epsilon \right)' \left( I - X (X'X)^{-1} X' \right) \left( \bar{Z}_\gamma + \epsilon \right),
$$

$$
\frac{1}{n} D'(P_W - P_X) D
$$

$$
= \frac{1}{n} \left( \bar{Z}_\gamma + X (\psi + \Lambda_{xx}^{-1} \Lambda_{xz}) + \epsilon \right)' \left( W (W'W)^{-1} W' - X (X'X)^{-1} X' \right) \left( \bar{Z}_\gamma + X (\psi + \Lambda_{xx}^{-1} \Lambda_{xz}) + \epsilon \right)
$$

$$
= \frac{1}{n} \left( \bar{Z}_\gamma + \epsilon \right)' \left( W (W'W)^{-1} W' - X (X'X)^{-1} X' \right) \left( \bar{Z}_\gamma + \epsilon \right)
$$

$$
= \frac{1}{n} \gamma' \bar{Z}' \left( I - X (X'X)^{-1} X' \right) \bar{Z}_\gamma + \frac{1}{n} \epsilon' \left( W (W'W)^{-1} W' - X (X'X)^{-1} X' \right) \epsilon
$$

$$
+ \frac{1}{n} \epsilon' \left( I - X (X'X)^{-1} X' \right) \bar{Z}_\gamma,
$$

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\[ \sqrt{\left( \frac{1}{n} D'(P_W - P_X)D \right)^{-1} - \left( \frac{1}{n} D'P_X D \right)^{-1}} = \sqrt{\frac{1}{n} D'(P_{X\perp} - (P_W - P_X)) D}{\left( \frac{1}{n} D'P_X D \right)(\frac{1}{n} D'P_X D)}. \]  

We also have \( \frac{1}{n} D'(P_{X\perp} - (P_W - P_X)) \epsilon = \frac{1}{n} \epsilon' \left( I - W (W'W)^{-1} W' \right) \epsilon, \) and \( \frac{1}{n} D'(P_{X\perp} - (P_W - P_X)) D = \frac{1}{n} \epsilon' \left( I - W (W'W)^{-1} W' \right) \epsilon. \) Define the following quantities,

\[ \omega_1 = \omega_1(n) = \frac{1}{n} \left( \bar{Z}_\gamma + \epsilon \right)' \left( I - X (X'X)^{-1} X' \right) \left( \bar{Z}_\gamma + \epsilon \right) - 1 \]
\[ \omega_2 = \omega_2(n) = \frac{1}{n} \epsilon' \left( I - X (X'X)^{-1} X' \right) \bar{Z}_\gamma - 1 \]
\[ \omega_3 = \omega_3(n) = \frac{1}{n} \epsilon' \left( W (W'W)^{-1} W' - X (X'X)^{-1} X' \right) \epsilon - 1 \]
\[ \omega_4 = \omega_4(n) = \frac{1}{n} \epsilon' \left( I - W (W'W)^{-1} W' \right) \epsilon - 1 \]
\[ \omega_5 = \omega_5(n) = \frac{1}{n} \gamma' \bar{Z}' \left( I - X (X'X)^{-1} X' \right) \epsilon. \]

Define the following events

\[ \mathcal{A}_1 = \left\{ |\omega_1| \leq 2 \sqrt{\frac{\log(n - p_x)}{n - p_x} + 2 \frac{\log(n - p_x)}{n - p_x}} \right\} \]
\[ \mathcal{A}_2 = \left\{ |\omega_2| \leq 2 \sqrt{\frac{\log(n - p_x)}{n - p_x} + 2 \frac{\log(n - p_x)}{n - p_x}} \right\} \]
\[ \mathcal{A}_3 = \left\{ |\omega_3| \leq 2 \sqrt{\frac{\log(p_x)}{p_x} + 2 \frac{\log(p_x)}{p_x}} \right\} \]
\[ \mathcal{A}_4 = \left\{ |\omega_4| \leq 2 \sqrt{\frac{\log(n - p)}{n - p} + 2 \frac{\log(n - p)}{n - p}} \right\} \]
\[ \mathcal{A}_5 = \left\{ |\omega_5| \leq C \sqrt{\frac{\log(n - p_x) \log(n - p_x)}{n}} \sqrt{\Sigma_{22} \cdot \gamma' \Lambda_{I|Ic} \gamma} \right\} \]

and define

\[ \mathcal{A} = \cap_{i=1}^{5} \mathcal{A}_i. \]
Conditioning on $X$, we have

$$\omega_1 + 1 \sim \frac{1}{n-p_x} \chi^2_{n-p_x} \quad \text{and} \quad \omega_2 + 1 \sim \frac{1}{n-p_x} \chi^2_{n-p_x}.$$ 

Conditioning on $W$, we have

$$\omega_3 + 1 \sim \frac{1}{p_z} \chi^2_{p_z} \quad \text{and} \quad \omega_4 + 1 \sim \frac{1}{n-p} \chi^2_{n-p}.$$

By Lemma 5, we can establish that

$$\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) \geq 1 - (\min\{n-p,p_z\})^{-c}. \quad (78)$$

Conditioning on $X$, $\omega_5 = \frac{1}{n} \sum_{i=1}^{n-p_x} U_i V_i$, where $U_i$ is independent of $V_i$, $U_i$ follows i.i.d normal with mean 0 and variance $\gamma' \Lambda_{I|X} \gamma$ and $V_i$ follows i.i.d normal with mean 0 and variance $\Sigma_{22}$. Note that $K = \|U_i V_i\|_{\psi_1} \leq 2 \sqrt{\gamma' \Lambda_{I|X} \gamma \sqrt{\Sigma_{22}}}$. By Lemma 6, we establish that

$$\mathbb{P}(A_5) \geq 1 - (n-p_x)^{-c}. \quad (79)$$

By the definition (75), we have the following expressions,

\[
\begin{align*}
\frac{1}{n} \epsilon' \left( W (W'W)^{-1} W' - X (X'X)^{-1} X' \right) \epsilon &= \frac{p_z}{n} \Sigma_{22} (1 + \omega_3), \\
\frac{1}{n} D' P_{X^\perp} D &= \frac{n-p_x}{n} \left( \gamma' \Lambda_{I|X} \gamma + \Sigma_{22} \right) (1 + \omega_1), \\
\frac{1}{n} D' (P_W - P_X) D &= \frac{n-p_x}{n} \left( \gamma' \Lambda_{I|X} \gamma \right) (1 + \omega_2) + \frac{p_z}{n} \Sigma_{22} (1 + \omega_3) + \omega_5, \\
\frac{1}{n} D' (P_{X^\perp} - (P_W - P_X)) \epsilon &= \frac{n-p}{n} \Sigma_{22} (1 + \omega_4), \\
\frac{1}{n} D' (P_{X^\perp} - (P_W - P_X)) D &= \frac{n-p}{n} \Sigma_{22} (1 + \omega_4).
\end{align*}
\]

By the first equation of (80), we have

$$L_{2,3} = L_{2,3}'(1 + \omega_4). \quad (81)$$
Define
\[ h_1 = h_1 = \frac{1}{n} D'(P_W - P_X)D \left( \frac{n - p_x}{n} (\gamma'I|I'\gamma) + \frac{p_z}{n} \Sigma_{22} \right) - 1. \tag{82} \]

Note that
\[ \frac{1}{n} D'(P_W - P_X)D = \frac{n - p_x}{n} (\gamma'I|I'\gamma) \left( 1 + \omega_2 + \frac{\omega_5}{n} \frac{n - p_x}{n} (\gamma'I|I'\gamma) \right) + \frac{p_z}{n} \Sigma_{22} (1 + \omega_3). \]

Hence, on the event \( A \),
\[ |h_1| \leq |\omega_2| + \left| \frac{n - p_x}{n} (\gamma'I|I'\gamma) \right| + |\omega_3| \leq C \left( \frac{\log p_x}{p_x} + \sqrt{\frac{\log(n - p_x)}{n - p_x}} \sqrt{\frac{\Sigma_{22}}{\gamma'I|I'\gamma}} + \frac{p_z}{n} \Sigma_{22} \right). \]

By plugging the second and fifth equation of (75) and (82) into (74), we have the following key expressions,
\[
\sqrt{\left( \frac{1}{n} D'(P_W - P_X)D \right)^{-1} - \left( \frac{1}{n} D'P_{X^\perp}D \right)^{-1}} = \sqrt{\frac{n - p_x}{n} \Sigma_{22} \left( \frac{n - p_x}{n} (\gamma'I|I'\gamma + \Sigma_{22}) \right) \left( \frac{n - p_x}{n} (\gamma'I|I'\gamma) + \frac{p_z}{n} \Sigma_{22} \right)} \sqrt{\frac{1 + \omega_4}{(1 + \omega_1)(1 + h_1)}}
\]
and hence
\[ L_{2,1} = L_{2,1}^* \sqrt{\frac{1 + \omega_4}{(1 + \omega_1)(1 + h_1)}} \tag{83} \]

Combined with the second and forth equation of (80), we have
\[
\frac{(\frac{1}{n} D'P_{X^\perp}D)^{-1} \frac{1}{n} D' (P_{X^\perp} - (P_W - P_X)) \epsilon}{\sqrt{(\frac{1}{n} D'(P_W - P_X)D)^{-1} - (\frac{1}{n} D'P_{X^\perp}D)^{-1}}} = \sqrt{\frac{1 - \frac{p_z}{n} \Sigma_{22} \left( \frac{\gamma'I|I'\gamma}{p_z \Sigma_{22}} + \frac{1}{n - p_x} \right)}{\gamma'I|I'\gamma \Sigma_{22} + \frac{1}{p_x}}} \sqrt{\frac{1 + \omega_1}{1 + \omega_4}}
\]
and hence
\[ L_{2,4} = L_{2,4}^* \sqrt{\frac{(1 + \omega_1)(1 + h_1)}{1 + \omega_4}} \tag{84} \]

By defining \( A \) as in (77), the control of terms (50) and (51) follows from (83), (81), (84) and (76), the control of probability (49) follows from (78) and (79).
5.2 Proof of Lemma 2

Note that conditioning on $W$, 
\[
\begin{pmatrix}
0 & v' \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
P_{v_1} & 0 \\
0 & P_{v_2}
\end{pmatrix}
\begin{pmatrix}
\Pi_1 \\
\Pi_2
\end{pmatrix}
\]
is jointly normal distribution. Since 
\[
\text{Cov}
\begin{pmatrix}
v' \Pi_1 \\
v' \Pi_2
\end{pmatrix}
, 
\begin{pmatrix}
P_{v_1} \Pi_1 \\
P_{v_2} \Pi_2
\end{pmatrix}
\mid W
\] = 0, we establish the independence. Hence, conditioning on $W$, we have 
\[
\begin{pmatrix}
\frac{1}{\|v_1\|_2} v' \Pi_1 \\
\frac{1}{\|v_2\|_2} v' \Pi_2
\end{pmatrix}
\perp 
\begin{pmatrix}
\frac{1}{\sqrt{n}} (\Pi_1' P_{v_1} \Pi_2 - (n - 1) \Theta_{12}) \\
\frac{1}{\sqrt{n}} (\Pi_2' P_{v_2} \Pi_2 - (n - 1) \Theta_{22})
\end{pmatrix},
\]
and hence establish (65). On $B_1 \cap B_4$, we have 
\[
\left| \frac{R_1}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \right| \leq C \left( \sqrt{s_{z_1} \log p} + \frac{1}{\|\gamma\|_2} \frac{s_{z_1} \log p}{\sqrt{n}} \right). \tag{85}
\]
On $B_8$, we have 
\[
\left| \frac{R_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \right| \leq C \frac{\log p}{\sqrt{n}}. \tag{86}
\]
On $B_2 \cap B_5$, we have 
\[
\left| \frac{R_3}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \right| \leq C \frac{\log p}{\sqrt{n}}. \tag{87}
\]
By (85), (86) and (87), we establish (66). Note that 
\[
\sqrt{\frac{\text{Var}(\hat{\Sigma}_{12})}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2}} - 1 = \frac{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2 - \Theta_{22}^2 \text{Var}_1 - \text{Var}_2}{\sqrt{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} \left( \sqrt{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} + \sqrt{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} \right)}
\]
and hence 
\[
\left| \sqrt{\frac{\text{Var}(\hat{\Sigma}_{12})}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2}} - 1 \right| \leq \frac{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2 - \Theta_{22}^2 \text{Var}_1 - \text{Var}_2}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} \left| \text{Var}_1 - \text{Var}_1 \right| \left( \frac{\text{Var}_1}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} + \frac{\text{Var}_2}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} \right)
\]
and hence 
\[
\left| \sqrt{\frac{\text{Var}(\hat{\Sigma}_{12})}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2}} - 1 \right| \leq \frac{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2 - \Theta_{22}^2 \text{Var}_1 - \text{Var}_2}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} \left| \text{Var}_1 - \text{Var}_1 \right| \left( \frac{\text{Var}_1}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} + \frac{\text{Var}_2}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2} \right) \tag{88}
\]
We have
\[ \left| \hat{\text{Var}}_2 - \text{Var}_2 \right| \leq C \max \left\{ \max_{1 \leq i,j \leq 2} \left| \hat{\Theta}_{ij} - \Theta_{ij} \right|, |\hat{\beta} - \beta| \right\} \]
and hence
\[ \frac{\left| \hat{\text{Var}}_2 - \text{Var}_2 \right|}{\hat{\Theta}^2_{22} \text{Var}_1 + \text{Var}_2} \leq C \sqrt{\frac{\log p}{n}}. \tag{89} \]
Also,
\[ \frac{\left| \hat{\Theta}^2_{22} - \Theta^2_{22} \right| \text{Var}_1}{\hat{\Theta}^2_{22} \text{Var}_1 + \text{Var}_2} \leq C \sqrt{\frac{\log p}{n}}. \tag{90} \]
It remains to control \( \frac{\left| \text{Var}_1 - \hat{\text{Var}}_1 \right|}{\hat{\Theta}^2_{22} \text{Var}_1 + \text{Var}_2} \), which is further upper bounded by
\[ \left| \frac{\text{Var}_1 - \hat{\text{Var}}_1}{\text{Var}_1} - 1 \right| = \left| \frac{\| \gamma \|_2^2}{\| \hat{\gamma} \|_2^2} \left\| \sum_{j \in \hat{V}} \hat{\gamma}_j \hat{V}_j \right\|_2^2 \hat{\Theta}_{11} + \beta^2 \hat{\Theta}_{22} - 2 \beta \hat{\Theta}_{12} - 1 \right| \tag{91} \]
By (149) and (150) in Guo et al. (2016b), we have
\[ \left| \frac{\| \gamma \|_2^2 - 1}{\| \hat{\gamma} \|_2^2} \right| \leq C \frac{1}{\| \hat{\gamma} \|_2^2} \left( s_{z_1} \frac{\log p_z}{n} + C s_{z_1} \left( s \frac{\log p}{n} \right)^2 + C \| \gamma \|_2 \sqrt{\frac{2 s_{z_1} \log p_z}{n}} \right), \tag{92} \]
and
\[ \left| \frac{\left\| \sum_{j \in \hat{V}} \hat{\gamma}_j \hat{V}_j \right\|_2^2 - 1}{\left\| \sum_{j \in V} \gamma_j \hat{V}_j \right\|_2^2} - 1 \right| \leq C s_{z_1} \sqrt{\frac{\log p}{n}}. \tag{93} \]
Note that
\[ \left| \frac{\hat{\Theta}_{11} + \beta^2 \hat{\Theta}_{22} - 2 \beta \hat{\Theta}_{12}}{\hat{\Theta}_{11} + \beta^2 \Theta_{22} - 2 \beta \Theta_{12}} - 1 \right| \leq C \max \left\{ \max_{1 \leq i,j \leq 2} \left| \hat{\Theta}_{ij} - \Theta_{ij} \right|, |\hat{\beta} - \beta| \right\} \leq C \sqrt{\frac{\log p}{n}} \left( 1 + \sqrt{s_{z_1}} \| \gamma \|_2 \right), \tag{94} \]
where the last inequality follows from the definition of \( B_2 \) and \( B_5 \) and the fact that \( \sqrt{\text{Var}_1} \leq C \sqrt{\| \gamma \|_2} \). Combining (91), (92), (93) and (94), we establish that
\[ \left| \frac{\text{Var}_1 - \hat{\text{Var}}_1}{\text{Var}_1} - 1 \right| \leq C \frac{1}{\| \hat{\gamma} \|_2^2} \left( s_{z_1} \frac{\log p_z}{n} + C s_{z_1} \left( s \frac{\log p}{n} \right)^2 + C \| \gamma \|_2 \sqrt{\frac{2 s_{z_1} \log p_z}{n}} \right) \]
\[ + C s_{z_1} \sqrt{\frac{\log p}{n}} + C \sqrt{\frac{\log p}{n}} \left( 1 + \sqrt{s_{z_1}} \| \gamma \|_2 \right) \leq C \frac{1}{\sqrt{s_{z_1} \log p}}. \tag{95} \]
where the second inequality follows from the assumption \( \|\gamma_0^*\|_2 \gg s_{z1} \log p / \sqrt{n} \) and \( \frac{\sqrt{s_{z1} \log p}}{n} \to 0 \). Combing (89), (90) and (95), we establish (67).

5.3 Proof of Lemma 3

In the following, we will show (69). The proof of (70) is similar and omitted here. By (66) and (2) in the main paper, on the event \( B \), we have

\[
z_{\alpha/2}(1 - g(n)) \leq z_{\alpha/2} \sqrt{\frac{\text{Var}(\hat{\Sigma}_{12})}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2}} - \frac{R_1 + R_2 + R_3}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \leq z_{\alpha/2}(1 + g(n)),
\]

where \( g(n) = C \left( \sqrt{s_{z1} \log p / \sqrt{n}} + \frac{1}{\|\gamma\|_2} \sqrt{s_{z1} \log p / \sqrt{n}} \right) \). Define the following events

\[
\mathcal{F}_0 = \left\{ \frac{M_1 + M_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + B(\theta, W) \geq z_{\alpha/2} \sqrt{\frac{\text{Var}(\hat{\Sigma}_{12})}{\Theta_{22}^2 \text{Var}_1 + \text{Var}_2}} - \frac{R_1 + R_2 + R_3}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} \right\}
\]

\[
\mathcal{F}_1 = \left\{ \frac{M_1 + M_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + B(\theta, W) \geq z_{\alpha/2}(1 - g(n)) \right\}
\]

\[
\mathcal{F}_2 = \left\{ \frac{M_1 + M_2}{\sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1}} + B(\theta, W) \geq z_{\alpha/2}(1 + g(n)) \right\}
\]

Note that \( \mathcal{F}_2 \cap B \subset \mathcal{F}_0 \cap B \subset \mathcal{F}_1 \cap B \). Hence, we have

\[
\left| P(\mathcal{F}_0) - (1 - EW\Phi(z_{\alpha/2} - B(\theta, W))) \right| \leq P(\mathcal{F}_0 \cap B^c)
\]

\[
+ \left| P(\mathcal{F}_0 \cap B) - (1 - EW\Phi(z_{\alpha/2} - B(\theta, W))) \right|
\]

\[
\leq P(\mathcal{F}_0 \cap B^c) + \max_{i=1,2} \left| P(\mathcal{F}_i \cap B) - (1 - EW\Phi(z_{\alpha/2} - B(\theta, W))) \right|
\]

Note that

\[
\left| P(\mathcal{F}_i \cap B) - (1 - EW\Phi(z_{\alpha/2} - B(\theta, W))) \right|
\]

\[
= \left| P(\mathcal{F}_i) + P(B) - P(\mathcal{F}_i \cup B) - (1 - \Phi(z_{\alpha/2} - B(\theta, W))) \right|
\]

\[
\leq \left| P(\mathcal{F}_i) - (1 - EW\Phi(z_{\alpha/2} - B(\theta, W))) \right| + |P(B) - P(\mathcal{F}_i \cup B)|.
\]
By (64), we have $P(\mathcal{F}_i \cup B) \to 1$ and $P(\mathcal{F}_0 \cap B^c) \to 0$ and hence it is sufficient to control the term $\max_{i=1,2} \left| P(\mathcal{F}_i) - (1 - EW \Phi (z_{\alpha/2} - B(\theta, W))) \right|$. In the following, we will focus on the case $i = 1$. The case for $i = 2$ is very similar. Note that

$$
\left| P(\mathcal{F}_1) - (1 - EW \Phi (z_{\alpha/2} - B(\theta, W))) \right| \leq EW \left| P(\mathcal{F}_1 | W) - (1 - \Phi (z_{\alpha/2} - B(\theta, W))) \right| + 2P(B^c 8) \tag{98}
$$

where the last inequality follows from the fact that

$$
\left| P(\mathcal{F}_1 | W) - (1 - \Phi (z_{\alpha/2} - B(\theta, W))) \right| \leq 2.
$$

Starting from here, we will separate the proof into three cases,

Case a. $\|\gamma_V\|_2 \ll \sqrt{n}$.

Case b. $\|\gamma_V\|_2 \geq c\sqrt{n}$ and $\sqrt{n} \Sigma_{12} \to \Delta_1^*$;

Case c. $\|\gamma_V\|_2 \geq c\sqrt{n}$ and $\sqrt{n} \Sigma_{12} \to \infty$;

Case a.

By (65), we have

$$
P(\mathcal{F}_1 | W) = E_{M_2 | W} P(\mathcal{F}_1 | M_2, W) = E_{M_2 | W} \left(1 - \Phi \left(\frac{z_{\alpha/2}(1 - g(n)) + B(\theta, W) \sqrt{\text{Var}_2 + \Theta_{22}^2 \text{Var}_1 - M_2}}{\sqrt{\Theta_{22}^2 \text{Var}_1}}\right)\right).
$$

By the proposition [KMT] in Mason et al. (2012), conditioning on $W$, there exist $\tilde{M}_2$ which has the same distribution with $M_2$ and $\tilde{M}_2 \sim N(0, \text{Var}_2)$ such that on the event $\mathcal{F}_4 = \left\{ \left| \tilde{M}_2 \right| \leq C_1 \sqrt{n} \right\}$, we have

$$
\left| \tilde{M}_2 - \tilde{M}_2 \right| \leq C_2 \left( \frac{\tilde{M}_2^2}{\sqrt{n} \text{Var}_2} + \sqrt{\frac{\text{Var}_2}{n}} \right).
$$
Hence we have

\[ P \left( F_1 \mid W \right) = E_{M_2 \mid W} \left( 1 - \Phi \left( \frac{z_{\alpha/2}(1 - g(n)) + B(\theta, W) \sqrt{\text{Var}_2 + \Theta^2_{22} \text{Var}_1 - \tilde{M}_2}}{\sqrt{\Theta^2_{22} \text{Var}_1}} \right) \right) \]

\[ = E_{\tilde{M}_2 \mid W} \left( 1 - \Phi \left( \frac{z_{\alpha/2}(1 - g(n)) + B(\theta, W) \sqrt{\text{Var}_2 + \Theta^2_{22} \text{Var}_1 - \tilde{M}_2}}{\sqrt{\Theta^2_{22} \text{Var}_1}} \right) \right) \]

\[ = E_{\tilde{M}_2 \mid W} \left( 1 - \Phi \left( \frac{z_{\alpha/2}(1 - g(n)) + B(\theta, W) \sqrt{\text{Var}_2 + \Theta^2_{22} \text{Var}_1 - \tilde{M}_2}}{\sqrt{\Theta^2_{22} \text{Var}_1}} \right) \right) + E_{\tilde{M}_2, \tilde{M}_2 \mid W} g_1(n), \]  

(99)

where

\[ g_1(n) = \Phi \left( \frac{z_{\alpha/2}(1 - g(n)) + B(\theta, W) \sqrt{\text{Var}_2 + \Theta^2_{22} \text{Var}_1 - \tilde{M}_2}}{\sqrt{\Theta^2_{22} \text{Var}_1}} \right) - \Phi \left( \frac{z_{\alpha/2}(1 - g(n)) + B(\theta, W) \sqrt{\text{Var}_2 + \Theta^2_{22} \text{Var}_1 - \tilde{M}_2}}{\sqrt{\Theta^2_{22} \text{Var}_1}} \right) \]

Note that

\[ 1 - \Phi \left( z_{\alpha/2} - B(\theta, W) \right) = E_{\tilde{M}_2 \mid W} \left( 1 - \Phi \left( \frac{z_{\alpha/2}(1 - g(n)) + B(\theta, W) \sqrt{\text{Var}_2 + \Theta^2_{22} \text{Var}_1 - \tilde{M}_2}}{\sqrt{\Theta^2_{22} \text{Var}_1}} \right) \right) \]

(100)

and

\[ \left| E_{\tilde{M}_2, \tilde{M}_2 \mid W} g_1(n) \right| \leq E_{\tilde{M}_2, \tilde{M}_2 \mid W} \left| g_1(n) \right| \cdot 1_{F_4} + E_{\tilde{M}_2, \tilde{M}_2 \mid W} \left| g_1(n) \right| \cdot 1_{F_4} \]

(101)

Note that

\[ E_{\tilde{M}_2, \tilde{M}_2 \mid W} \left| g_1(n) \right| \cdot 1_{F_4} \leq 2E_{\tilde{M}_2, \tilde{M}_2 \mid W} \left| \frac{\tilde{M}_2 - \tilde{N}_2}{\sqrt{\Theta^2_{22} \text{Var}_1}} \right| \]

(102)

By (99), (100), (101) and (102) lead to

\[ |P \left( F_1 \mid W \right) - \left( 1 - \Phi \left( z_{\alpha/2} - B(\theta, W) \right) \right)| \]

\[ \leq 2C_2 \frac{1}{\sqrt{\Theta^2_{22} \text{Var}_1}} E_{\tilde{M}_2, \tilde{M}_2 \mid W} \left( \frac{\tilde{M}_2^2}{\sqrt{n \text{Var}_2}} + \sqrt{\frac{\text{Var}_2}{n}} \right) \]

\[ + 2P \left( F_4^c \right) = 4C_2 \frac{1}{\sqrt{\Theta^2_{22} \text{Var}_1}} \sqrt{\frac{\text{Var}_2}{n}} + 2P \left( F_4^c \right), \]
where the last equality follows from the fact that \( E\tilde{M}_2^2 = \text{Var}_2 \). Combined with (98), we have

\[
\left| P(\mathcal{F}_1) - (1 - E_W \Phi (z_{\alpha/2} - B(\theta, W))) \right| \leq E_W \left( 4C_2 \frac{1}{\sqrt{\Theta_{22}^2 \text{Var}_1}} \sqrt{\frac{\text{Var}_2}{n}} \cdot 1_{B_8} \right) + 2P(\mathcal{F}_1^c) + 2P(B_8^c),
\]

where the last inequality follows from the definition of \( B_8 \). Under the assumption \( \|\gamma\|_2 \ll \sqrt{n} \), we show that \( \left| P(\mathcal{F}_1) - (1 - E_W \Phi (z_{\alpha/2} - B(\theta, W))) \right| \to 0 \). Combined with (96) and (97), we establish (69).

Case b.

Under the assumption \( \|\gamma\|_2 \geq c\sqrt{n} \), on the event \( B_8 \), we have \( \text{Var}_1 \to 0 \) and

\[
M_1 \overset{p}{\to} 0, \quad M_2 \overset{d}{\to} N(0, \text{Var}_2), \quad B(\theta, W) \to \frac{\Delta^*_1}{\sqrt{\text{Var}_2}}.
\]

By the bounded convergence theorem, we establish

\[
E_W \left| P(\mathcal{F}_1 \mid W) - \left( 1 - \Phi \left( z_{\alpha/2} - \frac{\Delta^*_1}{\sqrt{\text{Var}_2}} \right) \right) \right| \cdot 1_{B_8} \to 0
\]

and

\[
E_W \left| \Phi \left( z_{\alpha/2} - \frac{\Delta^*_1}{\sqrt{\text{Var}_2}} \right) - \Phi \left( z_{\alpha/2} - \frac{\Delta^*_1}{\sqrt{\text{Var}_2}} \right) \right| \cdot 1_{B_8} \to 0.
\]

By applying triangle inequality, we establish (69).

Case c.

This case is very similar to Case b. The only difference is \( B(\theta, W) \to \infty \) and on the event \( B_8 \), we have

\[
P(\mathcal{F}_1 \mid W) \cdot 1_{B_8} \to 1, \quad \Phi \left( z_{\alpha/2} - \frac{\Delta^*_1}{\sqrt{\text{Var}_2}} \right) \cdot 1_{B_8} \to 0.
\]

References


